Concerning Characterizations of Boundaries of Holomorphic 1-Chains within Complex Surfaces

by

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CHAPTER I

Background

For this section we assume X to be a complex manifold or an analytic variety of (complex) dimension n and that M is a closed, oriented, $\mathcal{C}^2$ real $(2p-1)$-chain in X. Let S be a holomorphic p-chain in $X \setminus \operatorname{spt} M$. We say M is the *boundary* of S (or that M *bounds* S) *within* X if the following three conditions hold.

- $[S]$, the current of integration of S, has a simple extension to X.
- $\operatorname{spt} S \Subset X$.
- $d[S] = [M]$ as currents (or $\partial S = M$ in the sense of Stokes) in X.

(Remark: Unless $M = 0$, the simple extension of $[S]$ is not d-closed in X and does not represent a holomorphic p-chain in X.) We may simply say that M bounds S, if X is clearly understood from the context. We refer to X as the *ambient space.*

Given X we wish to determine when M bounds a holomorphic chain within X. A procedure for the identification of such M is called a *characterization of boundaries of holomorphic chains within* X or for the sake of brevity, a *characterization within* X. In some literature, finding necessary and sufficient conditions to describe boundaries of holomorphic chains is referred to as the problème du bord.

Wermer [28] provides a characterization within $X = \mathbb{C}^n$ when M is a simple closed curve with stronger smoothness conditions. M bounds a holomorphic chain, in fact an analytic variety, if and only if M is a proper subset of its polynomial hull, which is equivalent to the vanishing of the polynomial moments of M (integrals over M of polynomial 1-forms in $\mathbb{C}^n$). In [15] Harvey and Lawson provide a characterization within any Stein space X. A closed, oriented, $\mathcal{C}^2$ real $(2p-1)$-chain M bounds a holomorphic p-chain if and only if all the moments of M (integrals over M of $\bar{\partial}$-closed $(p,p-1)$-forms of X) vanish. For $p>1$ they further give that M bounding a holomorphic chain is equivalent to the maximal complexity of M. In [16] Harvey and Lawson give a characterization within $X=\mathbb{CP}^n \backslash \mathbb{CP}^{n-q}$ for $q \leq p$ via moment conditions. When $q<p$ they give a characterization using maximal complexity.

Dolbeault and Henkin provide a characterization within $\mathbb{CP}^n$, for $p=1$ in [8] and for general p in [9]. Their characterization gives that M bounds a holomorphic chain if and only if M is maximally complex (vacuous when $p=1$) and a particular integral over M, parameterized by two vectorized variables ξ and η, somewhere locally agrees in the second derivatives of ξ with some linear combination of holomorphic solutions to a vectorized form of the shockwave equation $ff_\xi = f_\eta$. Connected work by El Kasimi provides a characterization within Grassmanian spaces when $p=1$ in [10] and some results for characterizations within certain compactifications of $\mathbb{C}^n$ in [11].

We remark that the characterization question for $p>1$ exhibits behavior differing from that for the case $p=1$. Maximal complexity is a local property and is a necessary condition for M to bound in any ambient space. But when $p=1$ maximal complexity of M vacuously holds. So maximal complexity is neither pertinent nor sufficient in the characterizations of holomorphic 1-chains. As this shows and thus cautions, results for $p>1$ may not immediately port to $p=1$. In reverse, results for $p=1$ tend to yield relevant insight for the case of $p>1$, whether by generalization of technique or by reduction (i.e. slicing) of the problem to $p=1$. A technique of slicing is employed for Dolbeault's and Henkin's result of a characterization within $\mathbb{CP}^n$ for general p in [9]. So the case $p = 1$ holds some special nuance though retaining relevance to the general characterization question.

In past work many questions concerning holomorphic chains have been reduced, by projection techniques, to the case of codimension 1. As relevant examples, we cite [18], [15], and [8]. This calls some attention to the case of holomorphic $(n-1)$-chains, also known as divisors.

Together these two considerations motivate a look at the case of simultaneous dimension 1 and codimension 1, which is $p = 1$ and $n = 2$. So this work is devoted to results concerning characterizations of holomorphic 1-chains within complex analytic surfaces.

CHAPTER II

Preliminaries

Here we present some basic definitions and properties concerning holomorphic chains and birational maps.

Let V be a subset of a complex manifold Z. We call V an *analytic variety* (also referred to as an analytic set) in Z if for all points $z \in Z$, there exists a neighborhood U of z in Z and analytic functions $\phi_1, \phi_2, \ldots \phi_\ell$ on U such that $V \cap U = \mathbb{V}(\phi_1, \phi_2, \ldots, \phi_\ell) \cap U$. (We use the notation $\mathbb{V}(\phi_1, \ldots, \phi_\ell)$ to represent the set where the functions $\phi_1, \phi_2, \ldots, \phi_\ell$ simultaneously vanish.) If V' is an analytic variety in Z that is also contained in V then V' is an analytic variety in V. Without any loss to understanding the previous concepts, we may also allow Z to be an analytic variety in a complex manifold. (For some of the basic definitions concerning analytic varieties, such as irreducibility, dimension, tangent cones, and intersections, see [13] pp.20-22, 60-65.)

A *holomorphic chain* (or analytic cycle) S in Z is a locally finite $\mathbb{Z}$-formal linear combination of analytic varieties in Z. S may be represented as $\sum_j \mu_j V_j$, where $\mu_j \in \mathbb{Z}$ and the V_j are distinct irreducible analytic varieties in Z. For an irreducible analytic varieties V_j, the corresponding μ_j is its *multiplicity*. (If an irreducible analytic variety is not contained in a summation representation, then it has multiplicity

0.) A *component* of S is an irreducible analytic variety having non-zero multiplicity. A holomorphic p-chain is a holomorphic chain having only p-dimensional components. We define the *support* of S, $\operatorname{spt} S$, to be the union of all the components of S. Note $\operatorname{spt} S$ is an analytic variety in Z. An analytic variety V may be identified with the holomorphic chain $\sum_j V_j$, where V_j are the irreducible components of V.

A holomorphic p-chain V determines a current of integration $[V]$. The currents of integrations of holomorphic p-chains may be characterized as locally rectifiable d-closed (p, p)-dimension currents having support with zero Hausdorff $(2p + 1)$ real-dimensional measure, [18].

Many of the properties of analytic varieties can be algebraically extended to holomorphic chains. Let $S = \sum_j \mu_j V_j$ and $T = \sum_k \nu_k W_k$ be two holomorphic chains in Z. If U is an open set in Z then the intersection of S with U (or the restriction of S to U) is given by $S \cap U = \sum_j \mu_j (V_j \cap U)$. The intersection of two holomorphic chains may be given as $S \cdot T = \sum_j \sum_k \mu_j \nu_k V_j \cdot W_k$. (Note: Two analytic varieties V and W also have a point-set notion of intersection denoted by $V \cap W$. Note $V \cdot W$ is a holomorphic chain having support $V \cap W$.)

If p is a point of intersection between analytic varieties V and W, we say it is a point of *locally transverse* intersection if for some neighborhood U about p, the components of $V \cap U$ transversely intersect the components of $W \cap U$. We say two holomorphic chains are *locally transverse* if each component of one intersects the components of the other locally transversally. (To see a difference between local transversality and "global" transversality, consider the intersection of $\mathbb{V}(z_2^2 - z_1^3 - z_1^2)$ and $\mathbb{V}(z_1)$ at $(0, 0)$. $\mathbb{V}(z_2^2 - z_1^3 - z_1^2)$ is irreducible and so has only one ("global") component with a singularity at $(0, 0)$. But it has two non-singular "local" components about $(0, 0)$ which transversally intersect $\mathbb{V}(z_1)$.)

An understanding of rational and birational maps will be needed for Chapter V. (Sources for much of the following include [13], [14], and [26].) Let X be an irreducible complex quasi-projective variety, that is a Zariski-open subset of a complex projective variety. X may be embedded in some complex projective space $\mathbb{CP}^m$, with homogeneous coordinates $[Z_0 : Z_1 : \cdots : Z_m]$. A rational function on X is defined as the quotient P/Q of two homogeneous polynomials P and Q in $Z_0, Z_1, \ldots, Z_m$ such that they have the same degree and $Q|_X \not\equiv 0$. Since X is irreducible, the collection of rational functions on X forms a ring. The function field of X, $\mathbb{C}(X)$, is defined as the quotient of the ring of rational functions on X by the subring of rational functions with numerators identically vanishing on X.

A rational map ϕ on X to $\mathbb{CP}^\ell$, denoted as $\phi : X - - \to \mathbb{CP}^\ell$, is a "map" given by $[\phi_0(x) : \phi_1(x) : \cdots : \phi_\ell(x)]$, where ϕ_j are rational functions on X and at least one doesn't identically vanish on X. By multiplying by a common denominator and eliminating common factors a rational map can also be represented as $[H_0(x) : H_1(x) : \cdots : H_\ell(x)]$, where the H_j are homogeneous polynomials of the same degree with no irreducible factor common to them all. A rational map is not a map in the true sense of the word. For there may exist points in X at which all the functions ϕ_j (or H_j) vanish. A standard example of this is the rational map $[Z_1 : Z_2]$ on $\mathbb{CP}^2$ (coord. $[Z_0 : Z_1 : Z_2]$), which is not defined at $[1 : 0 : 0]$.

Define $\mathcal{I}(\phi) = \mathbb{V}(H_0, H_1, \ldots, H_\ell)$ to be *indeterminacy set* of ϕ. As there is no factor common to all H_j the indeterminacy set will have codimension at least two. (Thus in complex surfaces the indeterminacy set is discrete.) If $\mathcal{I}(\phi) = \emptyset$, meaning the H_j never all simultaneously vanish, then we say ϕ is *regular* on X. Note that on $X \backslash \mathcal{I}(\phi)$, ϕ is regular and thus a well-defined holomorphic map. $X \backslash \mathcal{I}(\phi)$ is called the *domain of definition* of ϕ. From the previous definitions it holds that $\mathbb{C}(X)$ and

$\mathbb{C}(\overline{X})$ are equivalent. So in fact a rational map on X also defines a unique rational map on $\overline{X}$. Thus a rational map on X is fully determined by a rational map on any Zariski-open subset of X. So a rational map may be defined as a regular map on some Zariski-open subset of X. As a related result, a map defined and holomorphic on the complement of a variety of codimension at least two in X defines a rational map on X. (See [13], pg. 491.)

Define $\mathcal{C}(\phi)$, the *critical set* of ϕ, to be the closure of the critical set of ϕ on $X \backslash \mathcal{I}(\phi)$. Note $\mathcal{C}(\phi)$ is a subvariety of X.

As a rational map is not a true map, we need to be careful in discussing the notion of image and pre-image. To rigorously define these we first consider Γ_ϕ the graph of ϕ. We define Γ_ϕ to be the closure of the graph of ϕ over its domain of definition in the space $X \times \mathbb{CP}^\ell$. We also define the projections $\pi_1 : \Gamma_\phi \to X$ and $\pi_2 : \Gamma_\phi \to \mathbb{CP}^\ell$, which are regular maps. For a variety B in $\mathbb{CP}^\ell$ we define its *inverse image* to be $\pi_1(\pi_2^{-1}(B))$. For a variety A in X we define its *image* or *total transform through* ϕ to be $\pi_2(\pi_1^{-1}(A))$. We define the *proper transform through* ϕ of A as the closure in $\mathbb{CP}^\ell$ of the image (in the normal sense) of $A \cap (X \backslash \mathcal{I}(\phi))$. (One reference for total and proper transforms, in the case of codimension 1 holomorphic chains, also known as divisors, may be found in [13], pg.495.) If $Y \subseteq \mathbb{CP}^\ell$ contains the image of X, then we can say ϕ is a rational map from X to Y, denoted $\phi : X - - \to Y$.

If $\phi : X - - \to Y$ and $\psi : Y - - \to Z$ are two rational maps we can talk about their composition so long as the image of X is not contained in $\mathcal{I}(\psi)$. For then we can define $\psi \circ \phi$ by a regular map on $X \backslash (\mathcal{I}(\phi) \cup \phi^{-1}(\mathcal{I}(\psi)))$ to obtain a rational map $\psi \circ \phi : X - - \to Z$. So if ϕ is a inclusion of $X \subseteq Y$, then we can use this to define the restriction of the rational map ψ to X, so long as $X \not\subseteq \mathcal{I}(\psi)$. The image of $\psi|_X$ is an alternate definition for the proper transform of X under ψ.

Now if $\phi : X - - \rightarrow Y$ and $\psi : Y - - \rightarrow X$ are rational maps such that $\phi \circ \psi$ and $\psi \circ \phi$ are defined and equal the identity, then we say that X and Y are *birationally equivalent* and we call ϕ and ψ *birational* maps. Off of the indeterminacy set and the critical set ϕ is a biholomorphism between $X \backslash (\mathcal{I}(\phi) \cup \mathcal{C}(\phi))$ and $Y \backslash (\mathcal{I}(\phi^{-1}) \cup \mathcal{C}(\phi^{-1}))$. Note that $\Gamma_{\phi^{-1}}$ and Γ_{ϕ} are simply reflections of one another, and so taking the image under ϕ^{-1} does agrees with taking the pre-image under ϕ.

Since on its domain of definition a rational function is holomorphic, the image of $\mathcal{C}(\phi) \backslash \mathcal{I}(\phi)$ cannot intersect the domain of definition of ϕ^{-1}. (Examine the Jacobians.) Thus $\phi(\mathcal{C}(\phi)) \subseteq \mathcal{I}(\phi^{-1})$. Similarly $\phi^{-1}(\mathcal{C}(\phi^{-1})) \subseteq \mathcal{I}(\phi)$. A more useful statement can be made with complex surfaces. Namely, birational maps between complex surfaces may be factored as a composition of a sequence of blow-up and blow-down maps. As a reference see [13], pg.511.

CHAPTER III

Initial Discussion and Examples

To begin our look at characterizations within some non-Stein surfaces, we show some basic observations which serve to contrast the class of boundaries of holomorphic 1-chains in a Stein surface versus the class of boundaries of holomorphic 1-chains in either $\mathbb{CP}^2$ or $\mathbb{C} \times \mathbb{CP}^1$. We also issue some caveats, supported with examples, concerning how a boundary of a holomorphic chain may interact with the holomorphic chain itself.

One feature of boundaries of holomorphic chains in Stein spaces (not just surfaces) is that the holomorphic chain they bound is unique. For if M bounds two holomorphic chains S_1 and S_2, then $S_1 - S_2$ forms a holomorphic chain in $X \setminus \operatorname{spt} M$. This is d-closed as a current in X and by [18] this extends to a holomorphic chain in X with support contained in the closure of $\operatorname{spt} S_1 \cup \operatorname{spt} S_2$, which is relatively compact in X. By the maximum principle, the only compact holomorphic chains in a Stein space is the zero chain. But for spaces that contain compact analytic varieties (implying they are not Stein), the boundaries of holomorphic chains are not unique. For compact analytic varieties may be added to or subtracted from a holomorphic chain, leaving its boundary unchanged. But a "uniqueness" modulo the arbitrary addition of compact analytic varieties does hold, though sometimes veiled by some nuances in component bookkeeping. (For instance $\partial\Delta \times \{0\}$ bounds both $\Delta \times \{0\}$ and $-1 \cdot (\mathbb{CP}^1 \backslash \overline{\Delta}) \times \{0\}$ within $\mathbb{CP}^1 \times \mathbb{CP}^1$. The two holomorphic chains differ by $\mathbb{CP}^1 \times \{0\}$.)

Also by the characterization within Stein spaces given by Harvey and Lawson, [15], being a boundary of a holomorphic chain may be viewed as a closed condition, as it is equivalent to the vanishing of a collections of integrals over M. So in fact if we have a sequence of closed, oriented, $\mathcal{C}^2$ real $2p-1$-chains M_k that bound within X and $M_k \to M$, as currents, then it follows that M bounds within X.

Particularly the previous holds for boundaries of holomorphic 1-chains within $\mathbb{C}^2$. But a dramatically different arrangement occurs for $\mathbb{CP}^2$ and $\mathbb{C} \times \mathbb{CP}^1$. For one being a boundary of a holomorphic 1-chain in either of these latter spaces is not a closed condition. To illustrate this, we provide the following example.

Define $p_m(z)$ to be the Taylor polynomial of degree m which approximates e^z about $z = 0$. Then define γ_m to be the real 1-chain given consisting of one simple closed curve $\partial\Delta \to \mathbb{C}^2$ given by $\zeta \mapsto e^\zeta + p_m(\zeta^{-1})$. Define γ to be the real 1-chain consisting of one simple closed curve $\partial\Delta \to \mathbb{C}^2$ given by $\zeta \mapsto e^\zeta + e^{\zeta^{-1}}$. $\gamma_m \to \gamma$, as currents. Now define V_m to be the analytic variety (which may be defined in either $\mathbb{CP}^2 \backslash \operatorname{spt} \gamma_m$ or $\mathbb{C} \times \mathbb{CP}^1 \backslash \operatorname{spt} \gamma_m$) given as the graph of the meromorphic function $e^\zeta + p_m(\zeta^{-1})$ over the unit disk Δ. It holds that γ_m bounds V_m within $\mathbb{C} \times \mathbb{CP}^1$ (or $\mathbb{CP}^2$). However γ does not bound within $\mathbb{C} \times \mathbb{CP}^1$ (nor $\mathbb{CP}^2$).

In fact amongst closed, oriented, $\mathcal{C}^2$ real 1-chains in $\mathbb{C}^2$, those that bound within $\mathbb{CP}^2$ or $\mathbb{C} \times \hat{\mathbb{C}}$ are dense, when considering the topology of currents on their currents of integration. In particularly any closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2$, may be approximated by real 1-chains that bound. Note that it suffices to show this for a simple closed $\mathcal{C}^2$ real curve. We can make the strongest and easiest demonstration of this for $\mathbb{CP}^2$. Let γ be a closed, $\mathcal{C}^2$ real curve $\partial\Delta \to \mathbb{C}^2$ given by $\zeta \mapsto (f_1(\zeta), f_2(\zeta))$. On $\partial\Delta$, f_1 and f_2 can be approximated by functions of the form $\sum_{j=-J}^{\infty} c_j \zeta^j$. These approximating functions have meromorphic extensions on the unit disc, and so define a map from the unit disc into $\mathbb{CP}^2$. This gives an analytic variety in $\mathbb{CP}^2$ bounded by an approximate to γ.

For $\mathbb{C} \times \mathbb{CP}^1$, a demonstration of density is a little more involved. Again let γ be a simple closed $\mathcal{C}^2$ real curve. Define π_1 as the projection of $\mathbb{C} \times \mathbb{CP}^1$ onto the first coordinate. By an arbitrarily small perturbation of γ, we can assume $\pi_1\gamma$ has finitely many self-intersections in $\mathbb{C}$. Now by adding and subtracting "vertical" line segments above the self-intersections of $\pi_1\gamma$, decompose γ into a linear combination of simple closed $\mathcal{C}^0$ real curves having images under the projection π_1 that are simple curves bounding a domain with positive multiplicity in $\mathbb{C}$. Define these to be γ_j and so represent γ as $\sum_j \epsilon_j \gamma_j$, where each ϵ_j is ± 1. We may perturb each γ_j by an arbitrarily small amount to $\tilde{\gamma}_j$ which are represented as the graphs of $\mathcal{C}^2$ functions on the $\mathcal{C}^2$ boundary of a bounded, simply-connected domain in $\mathbb{C}$; the domain being denoted Ω_j. By further arbitrarily small perturbations we may assume these functions have an meromorphic extension to Ω_j. Using the graphs of these meromorphic functions over Ω_j, with multiplicity ϵ_j we derive a holomorphic 1-chain in $\mathbb{C} \times \mathbb{CP}^1$ which is bounded by a closed, oriented, $\mathcal{C}^2$ real 1-chain that arbitrarily approximates γ in the topology as understood through currents.

However it should be noted that approximation in some stronger sense may fail in $\mathbb{C} \times \mathbb{CP}^1$. For instance, if we had wished to approximate the γ above by a simple closed $\mathcal{C}^2$ real curve, this would not be possible for certain cases of γ. As a general class of example consider any simple closed $\mathcal{C}^2$ γ such that $\pi_1\gamma$ is a "figure-8" in $\mathbb{C}$. In particular, assume that $\pi_1\gamma$ has a single self-intersection at 0, winding number

$+1$ about 1 and winding number -1 about -1. (Such a curve could be built on the algebraic variety $\mathbb{V}(z_2^2 - z_1^2 + 1)$ to bound within $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$.) Any appropriately close approximate $\tilde{\gamma}$ of such a γ would still maintain a positive winding number about $z_1 = 1$ and a negative winding number about $z_1 = -1$. Suppose that $\tilde{\gamma}$ did bound some holomorphic 1-chain S within $\mathbb{C} \times \mathbb{CP}^1$. By the maximum principle it must be contained in $K \times \mathbb{CP}^1$, where K is the polynomial hull of $\pi_1(\tilde{\gamma})$. Then because S has no components over the unbounded component of $\mathbb{C} \backslash \pi_1(\tilde{\gamma})$ and by the winding numbers of $\tilde{\gamma}$, S must have a negative component over -1 and a positive component over 1. But as $\tilde{\gamma}$ is a simple curve in $\mathbb{C} \times \mathbb{CP}^1$, it only bounds one component, either with multiplicity $+1$ or -1. Hence this forms a contradiction.

The closed, oriented, $\mathcal{C}^2$ real 1-chains that bound within $\mathbb{CP}^2$ (or $\mathbb{C} \times \mathbb{CP}^1$) form a dense non-closed set within the entire set of closed, oriented, $\mathcal{C}^2$ real 1-chains. One product of the work of the following chapters, given in Chapter IX, is that this set of bounding closed, oriented, $\mathcal{C}^2$ real 1-chains may be given as an increasing union of closed sets.

These observations help provide an a priori acknowledgment that the characterizations within $\mathbb{CP}^2$ and $\mathbb{C} \times \mathbb{CP}^1$ must greatly differ from characterizations within $\mathbb{C}^2$. For one the characterizations within $\mathbb{CP}^2$ and $\mathbb{C} \times \mathbb{CP}^1$ will not involve the vanishing of moments. (In this context, a moment of a real 1-chain means the integration by a form over the given real 1-chain.) The lack of closedness prohibits a characterization involving only vanishing moments. But the property of density prohibits the presence on any non-vacuous moments in a characterization. So as we examine characterizations within $\mathbb{CP}^2$ and $\mathbb{C} \times \mathbb{CP}^1$, we should note in advance that their formulations must have a novel flavor.

Before proceeding, we wish to provide some cautionary remarks concerning the relation of the boundary of a holomorphic chain and the holomorphic chain itself.

As one note of caution, the notion of boundary with respect to currents or in the sense of Stokes can differ from some other notions of boundary. The following example, taken from [15], demonstrates a real-analytic chain M bounding a real-analytic submanifold S (as a holomorphic chain) within X while (S, M) is not a manifold with boundary in X.

Let $\Phi : \mathbb{C}^{n-1} \to \mathbb{C}^n$ be given by $(t, z_3, \ldots, z_n) \mapsto (t^2 - 1, t^3 - t, z_3, \ldots, z_n)$. Notably Φ is an immersion and has image $\mathbb{V}(z_2^2 - z_1^3 - z_1^2)$. $\Phi(-1, z_3, \ldots, z_n) = \Phi(1, z_3, \ldots, z_n) = (0, 0, z_3, \ldots, z_n)$. Otherwise Φ embeds $(\mathbb{C}\backslash\{-1, 1\}) \times \mathbb{C}^{n-2}$ into $(\mathbb{C}^2\backslash(0, 0)) \times \mathbb{C}^{n-2}$. Let B be the disc of radius 2 with center $a = (-1, 0, \ldots, 0)$ in $\mathbb{C}^{n-1}$. Let $b = (1, 0, \ldots, 0) \in \partial B$ and note $\Phi(a) = \Phi(b)$. $M = \Phi(\partial B)$ is real-analytic submanifold of $\mathbb{C}^n$ and $S = \Phi(B\backslash\{a\})$ is a real-analytic submanifold of $\mathbb{C}^n\backslash M$. M bounds S within $\mathbb{C}^n$, but (S, M) is not a submanifold with boundary.

The previous example and simple variations of it show that a boundary of a holomorphic chain can be intersected by a holomorphic chain it bounds. In slightly rougher language, we should be conscious of the possibility of the holomorphic chain "globally folding" back onto its boundary.

While the above shows we should perceive things in some good "local sense", one cannot attribute all issues to local properties. Global structure does still have a role.

For instance if γ is the graph of a continuous function f over a simple closed curve (enclosing some domain Ω), then it bounds within $\mathbb{C}^2$ if and only if f has a continuous extension to a function holomorphic over Ω. (See Theorem 20.2 of [1].) But this does not arise from local considerations. Namely if γ is a simple, closed, $\mathcal{C}^2$ real curve bounding within $\mathbb{C}^2$ (coord. (z_1, z_2)), then some portion of γ being the graph of a continuous (or even $\mathcal{C}^k$) function in z_1 does not imply there is a holomorphic 1-chain bounded by γ that nearby is the graph of a function in z_1.

As a concrete example, consider a portion of γ given as $(x, f(x))$ over a portion of the real axis, where $f(x) = x^{\frac{100}{3}}$. This portion of γ lies on the algebraic variety $\mathbb{V}(z_2^3 - z_1^{100})$, which is three-sheeted over $\mathbb{C}$. Locally it divides this variety into two portions. One portion consists of two sheets over a local portion of the half plane of positive imaginary part and one sheet over a local portion of the half plane of negative imaginary part. The other portion has the number of sheets over each local portion of half-plane reversed.

This local portion of a curve can be incorporated into a γ that bounds a holomorphic 1-chain, which would then contain one of the aforementioned portions of the given algebraic variety. However such γ must have self-intersections in the projection by π_1, due to the global result previously mentioned.

These examples may also be helpful test subjects for ruminating on basic properties and conjectures. We conclude our initial discussion and now begin our demonstration of characterization for some non-Stein spaces.

CHAPTER IV

The Dolbeault and Henkin Characterization within $\mathbb{CP}^2$

Dolbeault and Henkin provide a characterization of holomorphic 1-chains within $\mathbb{CP}^2$. This is the earliest example we found of a characterization within a non-Stein surface. This result is also the foundation of their more general results of characterizations of holomorphic p-chains in $\mathbb{CP}^n$. This section presents and elaborates upon their characterization within $\mathbb{CP}^2$.

For this section we use the following conventions of notation. Let γ be a closed, oriented $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \mathbb{CP}^2$. We denote homogeneous coordinates for $\mathbb{CP}^2$ as $(w_0 : w_1 : w_2)$ and affine coordinates for $\mathbb{C}^2 \subset \mathbb{CP}^2$ as $(z_1, z_2) = (w_1/w_0, w_2/w_0)$. Define $g = z_2 - \xi - \eta z_1$ and $\tilde{g} = w_2 - \xi w_0 - \eta w_1$. Then (ξ, η) serves as coordinates for an affine piece of $(\mathbb{CP}^2)'$. We also define the projection $\pi_\eta : \mathbb{C}^2 \to \mathbb{C}$ by $(z_1, z_2) \mapsto z_2 - \eta z_1$. Now for any (ξ, η) such that $\xi \notin \pi_\eta(\gamma)$ we define

$$G_\gamma(\xi, \eta) = \frac{1}{2\pi \mathrm{i}} \int_\gamma z_1 \frac{d(z_2 - \eta z_1)}{z_2 - \xi - \eta z_1} = \frac{1}{2\pi \mathrm{i}} \int_\gamma z_1 \frac{dg}{g}. \tag{4.1}$$

Here is a characterization of boundaries of holomorphic 1-chains within $\mathbb{CP}^2$ due to Dolbeault and Henkin.

Theorem IV.1. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \mathbb{CP}^2$, the following are equivalent:*

(i) γ *bounds a holomorphic 1-chain within* $\mathbb{CP}^2$

(ii) $\exists\ (\xi^*, \eta^*)$ *with some neighborhood* Ω *such that* $\exists$ *non-negative integers* N^+ *and* N^- *and functions* $f_j^+(\xi,\eta)$ *for* $1 \leq j \leq N^+$ *and* $f_j^-(\xi,\eta)$ *for* $1 \leq j \leq N^-$ *that are defined on* Ω*, analytic in* (ξ,η)*, and satisfy the shockwave equation,* $ff_\xi = f_\eta$*,* $(f = f_j^\pm)$*, such that on* Ω*,*

$$\frac{\partial^2}{\partial \xi^2} G_\gamma(\xi,\eta) = \frac{\partial^2}{\partial \xi^2}\left(\sum_{j=1}^{N^+} f_j^+(\xi,\eta) - \sum_{j=1}^{N^-} f_j^-(\xi,\eta)\right) \tag{4.2}$$

(Remark: This is Théorème II of [9] restricted to $p = 1$ and $n = 2$. If we modify condition (ii) by removing the second partial derivative symbols from (4.2) then the resulting theorem would also be valid. The theorem so modified is in fact the statement given in [8]. The statement of (ii), as we have given, may be more simply proven as equivalent to (i) and is a more natural condition for equivalence, as we shall later see.)

To prove Theorem IV.1 we use the same elements of the proof as given by Dolbeault and Henkin. However our arrangement will provide several stronger implications which have their own independent value. This will culminate in Theorem IV.6, which is a stronger and more structurally rich statement than Theorem IV.1.

First we provide a stronger form of (i) $\implies$ (ii), using the following definitions. Define $\mathcal{U}_\gamma = \{(\xi,\eta) \in \mathbb{C}^2 \mid \xi \notin \pi_\eta(\gamma)\}$, $\mathcal{T}_V = \{(\xi,\eta) \in \mathcal{U}_\gamma \mid \{w_2 = \xi w_0 + \eta w_1\}$ is not locally transverse to $V\}$, and $\mathcal{I}_V = \{(\xi,\eta) \in \mathcal{U}_\gamma \mid \{w_2 = \xi w_o + \eta w_1\} \cap V \cap \{w_0 = 0\} \neq \emptyset\}$.

Lemma IV.2. *Let* V *be a holomorphic 1-chain bounded by* γ *within* $\mathbb{CP}^2$ *and containing no components in the line at infinity* $\{w_0 = 0\}$*. For any simply-connected domain* $\Omega \subseteq \mathcal{U}_\gamma \backslash (\mathcal{T}_V \cup \mathcal{I}_V)$*, there exist nonnegative integers* N^+ *and* N^- *and functions*

$f_1^+, f_2^+, \ldots, f_{N^+}^+$ *and* $f_1^-, f_2^-, \ldots, f_{N^-}^-$ *well-defined, analytic in* (ξ, η)*, and satisfying the s.w. equation,* $ff_\xi = f_\eta$*, on* Ω*, for which (4.2) holds.*

Proof: As Ω is simply-connected and by continuation of analytic and constant functions, the proof need only be done locally. So let (ξ_0, η_0) be a point in $\mathcal{U}_\gamma \backslash (\mathcal{T}_V \cup \mathcal{I}_V)$ and let Ω be a simply-connected neighborhood of (ξ_0, η_0), which we may shrink as needed.

Let $\Omega_{\eta_0} = \{\xi \mid (\xi, \eta_0) \in \Omega\}$. Each component of $V \cap \pi_{\eta_0}^{-1}(\Omega_{\eta_0})$ forms an unbranched covering of Ω_{η_0} and is contained in $\mathbb{C}^2$. (Recall $\pi_\eta(z_1, z_2) = z_2 - \eta z_1$.) Similarly let $\Omega_{\xi_0} = \{\eta \mid (\xi_0, \eta) \in \Omega\}$. If $(0, \xi_0) \notin \operatorname{spt} V$, then each component of $V \cap \tau_{\xi_0}^{-1}(\Omega_{\xi_0})$ forms an unbranched covering of Ω_{ξ_0} and is contained in $\mathbb{C}^2$, where we define $\tau_\xi(z_1, z_2) = \frac{z_2 - \xi}{z_1}$.

For the present, assume V has no components contained in the line $\{w_1 = 0\}$. For $(\xi, \eta) \in \Omega$, define $N^+(\xi, \eta)$ and $N^-(\xi, \eta)$, respectively, as the positive and negative multiplicities of the intersections of the line $\{w_2 = \xi w_0 + \eta w_1\}$ with V. With the covering properties from the previous paragraph, we conclude that N^+ and N^- are locally constant, hence constant over Ω. These covering properties allow us to define $p_j^+(\xi, \eta)$ for $1 \leq j \leq N^+$ and $p_j^-(\xi, \eta)$ for $1 \leq j \leq N^-$ as analytic functions from Ω to $\mathbb{C}^2$ that give, respectively, the positive and negative points of intersection of the line $\{w_2 = \xi w_0 + \eta w_1\}$ with V. (Now if we allow V to have components contained in the line $w_1 = 0$, then the definitions for $N^\pm$ and $p_j^\pm(\xi, \eta)$ may be validly extended as given.) Define $f_j^\pm(\xi, \eta) = z_1|_{p_j^\pm(\xi, \eta)}$. Define q_s^+ for $1 \leq s \leq M^+$ and q_s^- for $1 \leq s \leq M^-$ to be the points of intersection between V and the line at infinity $\{w_0 = 0\}$, of positive and negative multiplicity, respectively.

Note

$$(4.3) \qquad G_\gamma(\xi,\eta) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{w_1}{w_0} \frac{d(\tilde{g}/w_0)}{\tilde{g}/w_0} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \left(\frac{w_1 d\tilde{g}}{w_0 \tilde{g}} - \frac{w_1 dw_0}{w_0^2} \right).$$

By residue calculations,

$$(4.4) \quad \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{w_1 d\tilde{g}}{w_0 \tilde{g}}$$
$$= \sum_j \frac{w_1}{w_0}\bigg|_{p_j^+(\xi,\eta)} - \sum_j \frac{w_1}{w_0}\bigg|_{p_j^-(\xi,\eta)} + \sum_s \frac{w_1}{\tilde{g}} \frac{d\tilde{g}}{dw_0}\bigg|_{q_s^+} - \sum_s \frac{w_1}{\tilde{g}} \frac{d\tilde{g}}{dw_0}\bigg|_{q_s^-}$$
$$= \sum_j f_j^+(\xi,\eta) - \sum_j f_j^-(\xi,\eta)$$
$$+ \sum_s \frac{w_1 \left(\frac{d(w_2 - \eta w_1)}{dw_0} - \xi \right)}{w_2 - \eta w_1} \Bigg|_{q_s^+} - \sum_s \frac{w_1 \left(\frac{d(w_2 - \eta w_1)}{dw_0} - \xi \right)}{w_2 - \eta w_1} \Bigg|_{q_s^-}.$$

By differentiating this equation twice with respect to ξ,

$$(4.5) \qquad \frac{\partial^2}{\partial \xi^2} \left(\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{w_1 d\tilde{g}}{w_0 \tilde{g}} \right) = \frac{\partial^2}{\partial \xi^2} \left(\sum_j f_j^+(\xi,\eta) - \sum_j f_j^-(\xi,\eta) \right).$$

Observe that $\int_\gamma \frac{w_1 dw_0}{w_0^2}$ is constant with respect to ξ. With the previous equations we see that (4.2) holds.

It only remains to be shown that each $f_j^\pm$ in fact satisfies the shockwave equation. While there are multiple means to achieve this, we present here a geometrical explanation. (In Dolbeault and Henkin a more formulaic demonstration is given, which they attribute to Darboux, [7].)

Let (ξ_0, η_0) be a point in Ω. Let f be one $f_j^\pm$, and p be the corresponding $p_j^\pm$. Let $h = f(\xi_0,\eta_0)$. By definition $p(\xi_0,\eta_0) = (h, \xi_0 + \eta_0 h)$ is contained in $W \cdot \{z_2 = \xi_0 + \eta_0 z_1\}$ for some component W of V. Therefore $p(\xi_0,\eta_0)$ is in $W \cdot \{z_2 = (\xi_0 - \tau h) + (\eta + \tau) z_1\}$ for τ such that $(\xi_0 - \tau h, \eta + \tau) \in \Omega$. For τ near 0, $p(\xi_0 - \tau h, \eta_0 + \tau) = p(\xi_0, \eta_0)$ and accordingly

$$(4.6) \qquad f(\xi_0 - \tau f(\xi_0,\eta_0), \eta_0 + \tau) = f(\xi_0,\eta_0).$$

(Remark: This says f is constant along lines of the form $(\xi - \xi_0) + f(\xi_0, \eta_0)(\eta - \eta_0) = 0$. In the terminology of partial differential equations, this prescribes a differential equation with characteristics of this form, along which solutions are constant.)

Equation (4.6) holds for all (ξ_0, η_0) in Ω for small τ. Differentiation with respect to τ and evaluation at $\tau = 0$ of (4.6) yields that

$$f_\xi(\xi_0, \eta_0)(-f(\xi_0, \eta_0)) + f_\eta(\xi_0, \eta_0) = 0. \tag{4.7}$$

This implies that f satisfies the s.w. partial differential equation on Ω.

$\square$

So a holomorphic 1-chain in $\mathbb{CP}^2$ has its behavior near the line $\{w_2 = \xi^* w_0 + \eta^* w_1\}$ "encoded" in the functions $f_j^\pm(\xi, \eta)$ near (ξ^*, η^*). We refer to the line $\{w_2 = \xi^* w_0 + \eta^* w_1\}$ as the *perspective line*, as this suggests its geometric role.

Since $\gamma \subseteq \mathbb{C}^2$, if V is a holomorphic 1-chain bounded by γ within $\mathbb{CP}^2$, then any component of V contained in the line at infinity, $\{w_0 = 0\}$, must be the entire analytic variety $\{w_0 = 0\}$. Subtracting these from V does change that γ bounds V. So if γ bounds within $\mathbb{CP}^2$, it will bound a holomorphic 1-chain with no components contained in the line at infinity.

(For ease of notation, we may drop the plus and minus superscripts, understanding the discussion equally applies to both. So f_j may be used in place of $f_j^\pm$.)

If the perspective line is locally transverse to V and does not intersect V at the line at infinity, then since these are open properties, Lemma IV.2 implies that condition (ii) of Theorem IV.1 holds for any appropriately small neighborhood of (ξ^*, η^*). With Theorem IV.1 in its present form, neither of these qualifications on the choice of the perspective line can be removed. Locally non-transverse intersections of the perspective line with V obstruct some of the f_j from being single-valued.

Intersections of V with the perspective line at infinity obstruct some f_j from being analytic due to an infinite limit at (ξ^*, η^*). We will address this further in Chapter VI.

Equation (4.2) may be interpreted as a decomposition of G_γ permitting a ξ-affine discrepancy. For notational brevity, we will call this a $\xi 2$-*decomposition* of G_γ. One value of Lemma IV.2 and its proof is that it associates holomorphic 1-chains bounded by γ to local $\xi 2$-decompositions of G_γ by s.w. solutions. To state this more technically, we introduce the following definitions.

Define HC_γ to be the $\mathbb{Z}$-affine space of holomorphic 1-chains that have no components contained in $\{w_0 = 0\}$ and that are bounded by γ within $\mathbb{CP}^2$. This is contained in the $\mathbb{Z}$-module (or lattice) of holomorphic chains in $\mathbb{CP}^2 \setminus \operatorname{spt} \gamma$. (We say H is $\mathbb{Z}$-affine if for any $c_j \in \mathbb{Z}$ such that $\sum_j c_j = 1$ and for any $v_j \in H$, then $\sum c_j v_j \in H$, or equivalently for some $v \in H$, $H - v$ is $\mathbb{Z}$-linear.) For $(\xi^*, \eta^*) \in \mathfrak{U}_\gamma$, define $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$ to be the $\mathbb{Z}$-affine subspace in HC_γ of all holomorphic 1-chains in HC_γ having only points of intersection with the line $\{w_2 = \xi^* w_0 + \eta^* w_1\}$ that are locally transverse and do not occur at the line at infinity.

Define $\mathrm{SW}_{(\xi^*,\eta^*)}$ to be the $\mathbb{Z}$-module of formal $\mathbb{Z}$-linear combinations of germs of analytic functions satisfying the s.w. equation $f f_\xi = f_\eta$ around (ξ^*, η^*). (This is also the free $\mathbb{Z}$-module with fore-said germs as formal generators). Define $\mathcal{O}_{(\xi^*,\eta^*)}$ to be the ring (also a $\mathbb{Z}$-module) of germs of analytic functions about (ξ^*, η^*). $\mathrm{SW}_{(\xi^*,\eta^*)}$ has a natural ($\mathbb{Z}$-module) homomorphism into $\mathcal{O}_{(\xi,\eta)}$ given by mapping each formal representative to the germ it symbolically represents.

We now pause to define the notions of formal and non-formal equivalence. Formal equivalence of two elements in $\mathrm{SW}_{(\xi^*,\eta^*)}$ is the usual equality understood among elements of $\mathrm{SW}_{(\xi^*,\eta^*)}$. Non-formal equivalence of two elements in $\mathrm{SW}_{(\xi^*,\eta^*)}$ is equivalence after application of the natural homomorphism into $\mathcal{O}_{(\xi^*,\eta^*)}$. To illustrate the difference, let's consider the following examples. Note that the constant functions are solutions to the shockwave equation. So $\mathbf{0}$, $\mathbf{1}$, and $\mathbf{2}$ are elements of $\mathrm{SW}_{(\xi^*,\eta^*)}$. $\mathbf{1}+\mathbf{1}-\mathbf{0}$ equals $\mathbf{2}$ non-formally, but not formally. The expression $\mathbf{1}+\mathbf{1}-\mathbf{0}$ could also be expressed as (and is formally equivalent to) $2\cdot\mathbf{1}+(-1)\cdot\mathbf{0}$. Also note that 0 (the zero of the formal ring) and $\mathbf{0}$ (the formal representative of the zero germ) are distinct formally though they are non-formally equal.

Define $\mathrm{SW}_{(\xi^*,\eta^*),G}$ to be the $\mathbb{Z}$-affine subspace of $\mathrm{SW}_{(\xi^*,\eta^*)}$ whose elements non-formally agree with G in the second derivative with respect to ξ. $\mathrm{SW}_{(\xi^*,\eta^*),G}$ signifies the space of $\xi 2$-decompositions of G about (ξ^*,η^*) by s.w. solutions.

We assume γ and (ξ^*,η^*) are fixed and define $\varphi : \mathrm{HC}_{\gamma,(\xi^*,\eta^*)} \to \mathrm{SW}_{(\xi^*,\eta^*),G_\gamma}$ to be the map prescribed by the proof of Lemma IV.2. It is a basic observation that φ is a $\mathbb{Z}$-affine map. Now our objective is to show that φ is an isomorphism.

Lemma IV.3. *φ is surjective.*

Proof: In [8], sections 3 and 4 regarding the *Condition suffisante dans* $\mathbb{CP}^2$, Dolbeault and Henkin provide a procedure for constructing a holomorphic 1-chain V bounded by γ from a decomposition $\sum_j f_j^+ - \sum_j f_j^-$ of G_γ. This procedure need only be slightly modified (with methods seen in [9]) to use a $\xi 2$-decomposition in place of an ordinary decomposition. (In particular Proposition 3.2 of [8] may be weakened by allowing P_m to be a polynomial in ξ of degree at most m (instead of $m-2$). This is comparable to Proposition 3.3.3 of [9].) As this procedure is rather lengthy we do not provide it here and instead leave the reader with the description above.

Define $\psi : \mathrm{SW}_{(\xi^*,\eta^*),G_\gamma} \longrightarrow \mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$ to be the map produced from this procedure. An examination of the procedure yields that $\varphi \circ \psi = \mathrm{id}$.

$\square$

Lemma IV.4. *φ is injective.*

Sublemma IV.5. Let V be an element of $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$. $\varphi(V) = 0$ if and only if $\operatorname{spt} V \cap \{w_2 = \xi^* w_0 + \eta^* w_1\} = \emptyset$.

Proof (of Sublemma): Let V be an element of $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$. Note $\varphi(V) = 0$ implies that the line $\{w_2 = \xi w_0 + \eta w_1\}$ doesn't intersect V for (ξ, η) near (ξ^*, η^*). Thus a open neighborhood of $\{w_2 = \xi^* w_0 + \eta^* w_1\}$ can be formed by a union of lines nearby in $(\mathbb{CP}^2)'$, none of which intersect V, thus $\operatorname{spt} V \cap \{w_2 = \xi^* w_0 + \eta^* w_1\} = \emptyset$.

The converse can be seen by reversing the previous argument.

$\square$

Proof (of Lemma): Suppose V_1 and V_2 are two holomorphic 1-chains in $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$ such that $\varphi(V_1) = \varphi(V_2)$. Let $V = V_1 - V_2$, which is a holomorphic 1-chain in $\mathbb{CP}^2 \setminus \operatorname{spt} \gamma$. Using Theorem 2.1 of [18], it holds V is a holomorphic 1-chain in $\mathbb{CP}^2$ and in particular is an element of $\mathrm{HC}_{\emptyset,(\xi^*,\eta^*)}$. Define $\varphi' : \mathrm{HC}_{\emptyset,(\xi^*,\eta^*)} \longrightarrow \mathrm{SW}_{(\xi^*,\eta^*),0}$ as the map φ for the case $\gamma = \emptyset$. Note $\varphi'(V) = \varphi(V_1) - \varphi(V_2) = 0$ (in the full formal sense).

By Sublemma IV.5, $\operatorname{spt} V$ avoids the line $w_2 = \xi^* w_0 + \eta^* w_1$. Thus V is a compactly supported holomorphic 1-chain in $\mathbb{CP}^2 \setminus \{w_2 = \xi^* w_0 + \eta^* w_1\}$. Therefore $V = 0$ since $\mathbb{CP}^2 \setminus \{w_2 = \xi^* w_0 + \eta^* w_1\}$ is Stein. (Given we are in $\mathbb{CP}^2$, this last step of the proof could alternately be achieved by applying Chow's Theorem and Bezout's theorem.)

□

Theorem IV.6. *The $\mathbb{Z}$-affine spaces* $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$ *and* $\mathrm{SW}_{(\xi^*,\eta^*),G_\gamma}$ *are isomorphic via the map* φ*.*

Proof: Combine Lemma IV.3 and Lemma IV.4.

□

(Remark: Theorem IV.1 may now be derived as a corollary to Theorem IV.6.)

For $(\xi^*,\eta^*) \in \mathcal{U}_\gamma$ the holomorphic 1-chains in $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$ can be isomorphically encoded (via φ) as the formal $\xi 2$-decompositions of G_γ by s.w. solutions near (ξ^*,η^*). As $p_j(\xi,\eta) = (f_j(\xi,\eta), \xi + \eta f_j(\xi,\eta))$, the formal $\xi 2$-decomposition will describe the structure of the corresponding holomorphic 1-chain near the perspective line. (This can also be seen as a consequence of Sublemma IV.5.) This is the essence of the isomorphism φ.

One implication of this is that the existence of a holomorphic 1-chain bounded by γ with some prescribed behavior near the perspective line directly correlates to the existence of a corresponding formal $\xi 2$-decomposition of G_γ by s.w. solutions locally about (ξ^*,η^*). By knowing how holomorphic 1-chain structure is encoded in s.w. equations, we may employ this correlation for further uses. The next section shows one immediate application of this principle in producing a characterization of boundaries of holomorphic 1-chains within $\mathbb{C} \times \mathbb{CP}^1$. (We will study this principle in fuller depth in Chapter VII.)

CHAPTER V

Characterizations within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ and $\mathbb{C} \times \hat{\mathbb{C}}$ via Birational Maps

The spaces $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ and $\mathbb{CP}^2$ are birationally equivalent. ($\hat{\mathbb{C}}$ denotes $\mathbb{CP}^1$.) One simple result of this is a characterization within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ derived from the characterization within $\mathbb{CP}^2$. But we can employ birational equivalence for more novel applications. In particular the previous section established a correspondence between the local behavior about a line of certain holomorphic 1-chains bounded by γ within $\mathbb{CP}^2$ and local formal s.w. $\xi 2$-decompositions of G_γ. Using this correspondence and the structure of a class of birational maps between $\mathbb{CP}^2$ and $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$, we produce the first characterization within $\mathbb{C} \times \hat{\mathbb{C}}$.

The first point we make is that birational maps treat the boundaries of holomorphic 1-chains reasonably well. Let X and Y be two complex surfaces. And let $\phi : X - - \to Y$ be a birational map. ϕ gives a biholomorphism between $X \backslash (\mathcal{I}(\phi) \cup \mathcal{C}(\phi))$ and $Y \backslash (\mathcal{I}(\phi^{-1}) \cup \mathcal{C}(\phi^{-1}))$. So away from the critical and indeterminacy sets ϕ_* provides a correspondence between ordinary topological chains. Holomorphic 1-chains on X can be mapped to holomorphic 1-chains on Y via either the total or proper transform, even if it should encounter the critical or indeterminacy set. Summarily the notion of the boundary of a holomorphic chain transforms well so long as the boundary (but not necessarily the holomorphic chain itself) avoids the critical and indeterminacy sets.

Theorem V.1. *Let X and Y be quasiprojective varieties of complex dimension 2 which are birationally equivalent via the map $\phi : X - - \to Y$. Let γ be a closed, oriented, $\mathcal{C}^2$ real 1-chain in $X \backslash (\mathcal{C}(\phi) \cup \mathcal{I}(\phi))$. (Note: $\phi_* \gamma$ is a closed, oriented, $\mathcal{C}^2$ real 1-chain in $Y \backslash (\mathcal{C}(\phi^{-1}) \cup \mathcal{I}(\phi^{-1}))$.) γ bounds within X if and only if $\phi_* \gamma$ bounds within Y.*

Proof: As $(\phi_*)^{-1} = (\phi^{-1})_*$, it suffices to prove the theorem in one direction. Suppose γ bounds a holomorphic chain S within X. We may locally define a holomorphic chain T in $Y \backslash \operatorname{spt}(\phi_* \gamma)$ by taking the proper transform (or the total transform) through ϕ of S near $\mathcal{C}(\phi) \cup \mathcal{I}(\phi)$. Away from the critical and indeterminacy sets, this agrees with $\phi_*(V)$, as here ϕ is biholomorphic. The simple extension of $[T]$ to Y and $d[T] = [\phi_* \gamma]$ will hold due to the simple extension of $[S]$ to X and $d[S] = [\gamma]$ by the local behavior of currents under biholomorphism. The set $\operatorname{spt} T$ is relatively compact in Y due to $\operatorname{spt} S$ being relatively compact in X (if using the total transform, then we additionally need the compactness of the exceptional divisors).

$\square$

While the total transform may be used for the above, our preference is for the proper transform, as this permits a simple formulation in terms of algebraic objects. For V an analytic variety in X avoiding $\operatorname{spt} \gamma$, define $\mathrm{HC}_\gamma^{X,V}$ to mean the $\mathbb{Z}$-module or lattice of holomorphic 1-chains that are bounded by γ within X and that have no components contained in V. For instance note $\mathrm{HC}_\gamma = \mathrm{HC}_\gamma^{\mathbb{CP}^2, \mathbb{V}(w_0)}$.

Theorem V.2. *Let X and Y be quasiprojective varieties of complex dimension 2 which are birationally equivalent via the map $\phi : X - - \to Y$. Let γ be a closed,* *oriented, $\mathcal{C}^2$ real 1-chain in $X\backslash(\mathcal{C}(\phi)\cup\mathcal{I}(\phi))$. Then* $\mathrm{HC}_{\gamma}^{X,\mathcal{C}(\phi)} \cong \mathrm{HC}_{\phi_*(\gamma)}^{Y,\mathcal{C}(\phi^{-1})}$ *via the proper transform through ϕ.*

Proof: Simply use the proof of the previous theorem. Only two items might prevent the proper transform through ϕ from being an isomorphism. One is that components contained in $\mathcal{C}(\phi)$ collapse to points in $\mathcal{I}(\phi^{-1})$. The other is that the proper transform never produces components in $\mathcal{C}(\phi^{-1})$ with non-zero multiplicity. But these are exactly the components we have excluded from consideration. For linear combinations of all other components, the proper transforms through ϕ and through ϕ^{-1} are inverses.

$\square$

Next we establish the following notation. Let $\hat{\mathbb{C}} = \mathbb{CP}^1$ and $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. For $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ we'll conventionally use homogeneous coordinates $(z_0 : z_1) \times (w_0 : w_1)$ and for $\mathbb{C}^2 \subset \hat{\mathbb{C}} \times \hat{\mathbb{C}}$, we use affine coordinates $(z, w) = (z_1/z_0, w_1/w_0)$. For $\mathbb{CP}^2$, we use homogeneous coordinates $(\breve{w}_0 : \breve{w}_1 : \breve{w}_2)$, with corresponding affine coordinates $(\breve{z}_1, \breve{z}_2)$ for $\mathbb{C}^2 \subset \mathbb{CP}^2$. This notation for $\mathbb{CP}^2$ is parallel to that used in the previous chapter.

It should be noted that $\mathbb{CP}^2$ and $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ are birationally equivalent. This can quite easily be seen by the birational map $\hat{\mathbb{C}} \times \hat{\mathbb{C}} -- \to \mathbb{CP}^2$ given by $(z_0 : z_1) \times (w_0 : w_1) \mapsto (z_0 w_0 : z_1 w_0 : z_0 w_1)$. This restricts to a map on affine piece $\mathbb{C}^2$, given as $(z, w) \mapsto (z, w)$. The critical set of this map is $\mathbb{V}(z_0 w_0)$ and the indeterminacy set is $\mathbb{V}(z_0, w_0)$. Note that the union of the critical and indeterminacy sets of the inverse map is the line at infinity. We derive the following corollary to Theorem V.2.

Corollary V.3. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2$,*

$$\mathrm{HC}_{\gamma}^{\hat{\mathbb{C}}\times\hat{\mathbb{C}},\mathbb{V}(z_0w_0)} \cong \mathrm{HC}_{\gamma}^{\mathbb{CP}^2,\mathbb{V}(\breve{w}_0)} = \mathrm{HC}_{\gamma}.$$

From this we give a characterization within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$.

Theorem V.4. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C} \times \mathbb{C} \subset \hat{\mathbb{C}} \times \hat{\mathbb{C}}$, the following are equivalent:*

1. *γ bounds a holomorphic 1-chain within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$*

2. *(condition (ii) of Theorem IV.1)*

   $\exists$ (ξ^*, η^*) with some neighborhood Ω such that $\exists$ integers N^+ and N^- and functions $f_j^+(\xi,\eta)$ for $1 \leq j \leq N^+$ and $f_j^-(\xi,\eta)$ for $1 \leq j \leq N^-$ that are defined on Ω, analytic in (ξ,η), and satisfy the shockwave equation, $f f_\xi = f_\eta$, ($f = f_j^\pm$), such that on Ω,

$$\frac{\partial^2}{\partial \xi^2} G_\gamma(\xi,\eta) = \frac{\partial^2}{\partial \xi^2}\left(\sum_{j=1}^{N^+} f_j^+(\xi,\eta) - \sum_{j=1}^{N^-} f_j^-(\xi,\eta)\right), \tag{5.1}$$

   where G_γ is given by (4.1).

Proof: Use the previous corollary and apply Theorem IV.1.

$\square$

To produce a characterization within $\mathbb{C} \times \hat{\mathbb{C}}$, we examine a class of birational maps between $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ and $\mathbb{CP}^2$. Define $\phi_\upsilon : \hat{\mathbb{C}} \times \hat{\mathbb{C}} - - \to \mathbb{CP}^2$ by $\phi_\upsilon : (z_0 : z_1) \times (w_0 : w_1) \mapsto (z_0 w_1 : z_1(w_0 - \upsilon w_1) : z_0(w_0 - \upsilon w_1))$. On the natural affine pieces of these spaces the map ϕ can be expressed in non-homogeneous coordinates as $\phi_\upsilon|_{\mathbb{C}^2} : (z, w) \mapsto (z(\frac{1}{w} - \upsilon), \frac{1}{w} - \upsilon)$. This can be seen to be a birational map. The inverse map can be given as $\phi_\upsilon^{-1} : \mathbb{CP}^2 - - \to \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ with $\phi_\upsilon^{-1} : (\breve{w}_0 : \breve{w}_1 : \breve{w}_2) \mapsto$

$(\breve{w}_2 : \breve{w}_1) \times (\breve{w}_2 + \upsilon\breve{w}_0 : \breve{w}_0)$. The inverse map can also be defined by the map on the affine $\mathbb{C}^2$ in $\mathbb{CP}^2$ given as $\phi_\upsilon^{-1}|_{\mathbb{C}^2} : (\breve{z}_1, \breve{z}_2) \mapsto (\frac{\breve{z}_1}{\breve{z}_2}, \frac{1}{\breve{z}_2+\upsilon})$.

We point out several of the properties concerning ϕ. The lines $\{z_0 = 0\}$ and $\{w_0 = \upsilon w_1\}$ blow down to the points $(0 : 1 : 0)$ and $(1 : 0 : 0)$, respectively. The point $(0 : 1) \times (\upsilon : 1)$ blows up to the line $\{\breve{w}_2 = 0\}$ in $\mathbb{CP}^2$. The map ϕ may be factored as a composition of a blow-up at $(0 : 1) \times (\upsilon : 1)$ followed by two blow-downs along the proper transforms (through the blow-up) of $\mathbb{V}(z_0)$ and $\mathbb{V}(w_0 - \upsilon w_1)$. Note $\mathcal{C}(\phi_\upsilon) = \mathbb{V}(z_0(w_0 - \upsilon w_1))$, $\mathcal{I}(\phi_\upsilon) = (0 : 1) \times (\upsilon : 1)$, $\mathcal{C}(\phi_\upsilon^{-1}) = \mathbb{V}(\breve{w}_2)$, and $\mathcal{I}(\phi_\upsilon^{-1}) = \{(0 : 1 : 0), (1 : 0 : 0)\}$.

We now discuss the concepts of tangential and non-tangential contact. Let V and W be two analytic varieties of dimension 1 (or for that matter, two holomorphic 1-chains) in a space of complex dimension two. Suppose that V and W intersect at a point p. We say V and W intersect with *non-tangential contact* if the tangent cones of V and W intersect trivially, that is at their zero point. Likewise we say V and W intersect with *tangential contact* if their tangent cones intersect non-trivially, that is, they contain a common complex line. Note that a locally transverse intersection implies an intersection of non-tangential contact, but not conversely. For instance in $\mathbb{C}^2$, $\mathbb{V}(w^2 - z^3)$ and $\mathbb{V}(z)$ intersect at $(0, 0)$ with non-tangential contact, but not locally transversally.

We return our attention to ϕ_υ. Let V be a local piece of analytic variety and W its proper transform through ϕ_υ. Away from $\mathcal{C}(\phi_\upsilon) \cup \mathcal{I}(\phi_\upsilon) = \mathbb{V}(z_0(w_0 - \upsilon w_1))$, the biholomorphic nature of ϕ_υ can be used to easily understand the transform of V to W. Near $\mathbb{V}(z_0(w_0 - \upsilon w_1))$, we need to examine the birational structure of ϕ_υ to understand the nature of its proper transform.

Recall our previous factorization of ϕ_υ into a blow-up and two blow-downs. From this we deduce the following statements. V intersects the line $\mathbb{V}(w_0 - \upsilon w_1)$ away from $(0:1)\times(\upsilon:1)$ if and only if W intersects the line $\mathbb{V}(\breve{w}_2)$ at $(1:0:0)$ with non-tangential contact. Similarly, V intersects the line $\mathbb{V}(z_0)$ away from $(0:1)\times(\upsilon:1)$ if and only if W intersects the line $\mathbb{V}(\breve{w}_2)$ at $(0:1:0)$ with non-tangential contact. V intersects $\mathbb{V}(w_0-\upsilon w_1)$ with tangential contact at $(0:1)\times(\upsilon:1)$ if and only if W intersects $\mathbb{V}(\breve{w}_2)$ at $(1:0:0)$ with tangential contact. In parallel fashion, V intersects $\mathbb{V}(z_0)$ with tangential contact at $(0:1)\times(\upsilon:1)$ if and only if W intersects $\mathbb{V}(\breve{w}_2)$ at $(0:1:0)$ with tangential contact. V intersects both $\mathbb{V}(w_0-\upsilon w_1)$ and $\mathbb{V}(z_0)$ (or simply $\mathbb{V}(z_0(w_0-\upsilon w_1))$) at $(0:1)\times(\upsilon:1)$ with non-tangential contact if and only if W intersects $\mathbb{V}(\breve{w}_2)$ neither at $(1:0:0)$ nor $(0:1:0)$. These statements exhaustively categorize the behavior of V near $\mathbb{V}(z_0(w_0-\upsilon w_1))$ and W near $\mathbb{V}(\breve{w}_2)$.

We encode these observations in terms of algebraic objects. First Theorem V.2 implies that

$$\mathrm{HC}_{\gamma}^{\hat{\mathbb{C}}\times\hat{\mathbb{C}},\mathbb{V}(z_0(w_0-\upsilon w_1))} \cong \mathrm{HC}_{(\phi_\upsilon)_*\gamma}^{\mathbb{CP}^2,\mathbb{V}(\breve{w}_2)} \tag{5.2}$$

via the proper transform due to ϕ_υ.

Next we confine the holomorphic 1-chains being considered to use of the scope of our knowledge, which is in HC_γ. Thus we remove from the right hand object any chains containing components in the line at infinity, and to preserve an isomorphism, we remove from the left hand object any chains with components in $\mathbb{V}(w_1)$, the pre-image of $\mathbb{V}(\breve{w}_0)$ under ϕ_υ. Thus

$$\mathrm{HC}_{\gamma}^{\hat{\mathbb{C}}\times\hat{\mathbb{C}},\mathbb{V}(z_0 w_1(w_0-\upsilon w_1))} \cong \mathrm{HC}_{(\phi_\upsilon)_*\gamma}^{\mathbb{CP}^2,\mathbb{V}(\breve{w}_0\breve{w}_2)}. \tag{5.3}$$

Now we restrict the left side to the holomorphic 1-chain bounded by γ within $\mathbb{C}\times\hat{\mathbb{C}}$. This will yield $\mathrm{HC}_{\gamma}^{\mathbb{C}\times\hat{\mathbb{C}},\mathbb{V}(w_1(w_0-\upsilon w_1))}$ on the left. Define HC'_γ to be the $\mathbb{Z}$-affine space of all holomorphic 1-chains bounded by γ within $\mathbb{CP}^2$ that contain no components in $\mathbb{V}(\breve{w}_0)$ and that only intersect $\mathbb{V}(\breve{w}_2)$ with non-tangential contact at $(1:0:0)$ (an empty intersection qualifies). By the previous observations concerning the behavior of holomorphic chains near $\mathbb{V}(z_0(w_0-\upsilon w_1))$, this is the correspondent to $\mathrm{HC}_\gamma^{\mathbb{C}\times\hat{\mathbb{C}},\mathbb{V}(w_1(w_0-\upsilon w_1))}$ within the previous isomorphism. Explicitly stated,

$$\mathrm{HC}_\gamma^{\mathbb{C}\times\hat{\mathbb{C}},\mathbb{V}(w_1(w_0-\upsilon w_1))} \cong \mathrm{HC}'_{(\phi_\upsilon)_*\gamma}. \tag{5.4}$$

If we had direct knowledge of HC_γ we could halt here, but we need to be within $\mathrm{HC}_{\gamma,(0,0)}$ to make any use of Theorem IV.6. The only qualification that remains to be satisfied is that our holomorphic 1-chains in $\mathbb{CP}^2$ need to intersect the line $\{\breve{w}_2 = 0\}$ locally transversally. Non-tangential contact does not imply this. However local transverse intersections with $\mathbb{V}(w_0 - \upsilon w_1)$ equates with local transverse intersections with $\mathbb{V}(\breve{w}_2)$ at $(1:0:0)$ through the proper transform. So define $\mathrm{HC}_{\gamma,\upsilon}^{\mathbb{C}\times\hat{\mathbb{C}}}$ to consist of all holomorphic 1-chains bounded by γ within $\mathbb{C}\times\hat{\mathbb{C}}$ that contain no components in $\mathbb{V}(w_1)$ and which are locally transverse to $\mathbb{V}(w_0-\upsilon w_1)$. And define HC''_γ to be all holomorphic 1-chains bounded by γ within $\mathbb{CP}^2$ that contain no components in $\mathbb{V}(\breve{w}_0)$ and that have only locally transverse intersections with $\mathbb{V}(\breve{w}_2)$ at $(1:0:0)$. We may restrict the above isomorphism to our working isomorphism, namely

$$\mathrm{HC}_{\gamma,\upsilon}^{\mathbb{C}\times\hat{\mathbb{C}}} \cong \mathrm{HC}''_{(\phi_\upsilon)_*\gamma}. \tag{5.5}$$

We can recapitulate the previous isomorphisms and inclusions in the following.

$$\begin{tikzcd}
\mathrm{HC}_{\gamma,\upsilon}^{\mathbb{C}\times\hat{\mathbb{C}}} \arrow[r, "\subset"] \arrow[d, "\cong"'] & \mathrm{HC}_\gamma^{\mathbb{C}\times\hat{\mathbb{C}},\mathbb{V}(w_1(w_0-\upsilon w_1))} \arrow[r, "\subset"] \arrow[d, "\cong"'] & \mathrm{HC}_\gamma^{\hat{\mathbb{C}}\times\hat{\mathbb{C}},\mathbb{V}(z_0 w_1(w_0-\upsilon w_1))} \arrow[d, "\cong"'] \\
\mathrm{HC}''_{(\phi_\upsilon)_*\gamma} \arrow[r, "\subset"] & \mathrm{HC}'_{(\phi_\upsilon)_*\gamma} \arrow[r, "\subset"] & \mathrm{HC}_{(\phi_\upsilon)_*\gamma}^{\mathbb{CP}^2,\mathbb{V}(\breve{w}_0\breve{w}_2)}
\end{tikzcd} \tag{5.6}$$

Set $(\xi^*,\eta^*) = (0,0)$, consider υ fixed, and let $\gamma' = (\phi_\upsilon)_*\gamma$. Using Theorem IV.6, note $\mathrm{HC}''_{\gamma'} \subset \mathrm{HC}_{\gamma',(0,0)} \overset{\varphi}{\cong} \mathrm{SW}_{(0,0),G_{\gamma'}}$. So we now reveal a description for $\varphi(\mathrm{HC}''_{\gamma'})$.

Let $V \in \mathrm{HC}''_{\gamma'}$. By local transversality of the intersection of V between $\{\breve{w}_2 = 0\}$, V is locally (near $\{w_2 = 0\}$) the formal linear combination of analytic varieties, each intersecting $\{\breve{w}_2 = 0\}$ transversely at $(1:0:0)$. Let W be one of these local portions with a corresponding shockwave solution given as $f(\xi, \eta)$. Then for η close enough to 0, each line of the form $\{\breve{w}_2 = \eta\breve{w}_1\}$ will intersect W transversally and at $(1:0:0)$. Recognizing the construction of the shockwaves in Lemma IV.2, we conclude that $f(\xi, \eta)$ must vanish along $\xi = 0$. (Alternately we could note that $f(0,0) = 0$ and use that shockwave solutions are constant along its characteristic lines to show that f must locally vanish along $\xi = 0$.)

Let $\mathrm{SW}''_{(0,0)}$ denote the formal $\mathbb{Z}$-linear combinations of local shockwave solutions which vanish along $\xi = 0$. Let $\mathrm{SW}''_{(0,0),G}$ denote the $\mathbb{Z}$ affine subspace of those which non-formally agree with G in the second derivative with respect to ξ. What we said previously may be stated as

$$\mathrm{HC}''_{\gamma'} \overset{\varphi}{\cong} \mathrm{SW}''_{(0,0),G_{\gamma'}} \tag{5.7}$$

So with (5.5) we have that

$$\mathrm{HC}^{\mathbb{C}\times\hat{\mathbb{C}}}_{\gamma,\upsilon} \cong \mathrm{HC}''_{(\phi_\upsilon)_*\gamma} \cong \mathrm{SW}''_{(0,0),G_{(\phi_\upsilon)_*\gamma}}. \tag{5.8}$$

From this, we now state the following characterization of boundaries of holomorphic chains in $\mathbb{C} \times \hat{\mathbb{C}}$.

Theorem V.5. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C} \times \mathbb{C}^* \subset \mathbb{C} \times \hat{\mathbb{C}}$, the following are equivalent:*

1. *γ bounds a holomorphic 1-chain within $\mathbb{C} \times \hat{\mathbb{C}}$*
2. *$\exists\ \upsilon$ with $\{w_0 = \upsilon w_1\} \cap \operatorname{spt}\gamma = \emptyset$ and a neighborhood Ω (with coordinates (ξ, η)) of $(0,0)$ such that $\exists$ integers N^+ and N^- and functions $f_j^+(\xi, \eta)$ for $1 \le j \le N^+$*

and $f_j^-(\xi,\eta)$ for $1 \le j \le N^-$ that are defined on Ω, analytic in (ξ,η), satisfy the shockwave equation, $ff_\xi = f_\eta$, $(f = f_j^\pm)$, and vanish when $\xi = 0$ such that on Ω,

(5.9)
$$\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{z}{\frac{w}{1-\upsilon w}}\frac{d(1-\xi\frac{w}{1-\upsilon w}-\eta z)}{1-\xi\frac{w}{1-\upsilon w}-\eta z}\right) = \frac{\partial^2}{\partial\xi^2}\left[\sum_{j=1}^{N^+} f_j^+(\xi,\eta) - \sum_{j=1}^{N^-} f_j^-(\xi,\eta)\right]$$

Proof: Condition 1 is equivalent to $\mathrm{HC}_\gamma^{\mathbb{C}\times\hat{\mathbb{C}},\emptyset}$ being nonempty. Since γ lies in $\mathbb{C}\times\mathbb{C}^*$, this is equivalent to $\mathrm{HC}_\gamma^{\mathbb{C}\times\hat{\mathbb{C}},\mathbb{V}(w_1(w_0-\upsilon w_1))}$ being nonempty for any υ such that $\{w_0 = \upsilon w_1\} \cap \operatorname{spt}\gamma = \emptyset$. This is in turn equivalent to the non-emptiness of $\mathrm{HC}_{\gamma,\upsilon}^{\mathbb{C}\times\hat{\mathbb{C}}}$ for a generic choice of such υ. This is equivalent to condition 2 due to (5.8) and the following calculation.

$$\begin{aligned}(5.10)\quad \frac{\partial^2}{\partial\xi^2}G_{\phi_*(\gamma)} &= \frac{\partial^2}{\partial\xi^2}\left(\frac{1}{2\pi\mathrm{i}}\int_\gamma z(\frac{1}{w}-\upsilon)\frac{d((\frac{1}{w}-\upsilon)-\xi-\eta z(\frac{1}{w}-\upsilon))}{(\frac{1}{w}-\upsilon)-\xi-\eta z(\frac{1}{w}-\upsilon)}\right)\\ &= \frac{\partial^2}{\partial\xi^2}\left(\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{z(1-\upsilon w)}{w}\frac{d(1-\xi\frac{w}{1-\upsilon w}-\eta z)}{1-\xi\frac{w}{1-\upsilon w}-\eta z} + \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{z(1-\upsilon w)}{w}\frac{d(\frac{1-\upsilon w}{w})}{\frac{1-\upsilon w}{w}}\right)\\ &= \frac{\partial^2}{\partial\xi^2}\left(\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{z}{\frac{w}{1-\upsilon w}}\frac{d(1-\xi\frac{w}{1-\upsilon w}-\eta z)}{1-\xi\frac{w}{1-\upsilon w}-\eta z}\right).\end{aligned}$$

$\square$

Remark: After the results of the coming chapter, we can pose theorems that allow us to fix a value of υ, rather than handling it generically. Specifically Theorem VII.4 is a corresponding result with υ fixed as zero.

CHAPTER VI

Other Characterizations within $\mathbb{CP}^2$

We now return to the study of characterizations within $\mathbb{CP}^2$. We present some additional characterizations that provide some improvements upon Theorem IV.1 and Theorem IV.6.

One weakness of Theorem IV.1 is that it does not permit the perspective line $\{w_2 = \xi^* w_0 + \eta^* w_1\}$ to be a priori fixed for testing condition (ii). We see from Lemma IV.2 that if γ bounds some V within $\mathbb{CP}^2$, then any choice of (ξ^*, η^*) in $\mathcal{U}_\gamma \backslash (\mathcal{T}_V \cup \mathcal{I}_V)$ would yield a successful validation of (ii). But having foreknowledge of a V bounded by γ also precludes the point of using (ii) to establish (i). As it stands, there is not an apparent way to determine solely from γ as to which (ξ^*, η^*) may be used to satisfy (ii). Significantly, demonstrating failure of (ii) requires showing the absence of a s.w. ξ2-decomposition of G_γ about a "substantial" set of (ξ^*, η^*) in $\mathcal{U}_\gamma$. (With simply the statement of Theorem IV.1 this substantial set must be all (ξ^*, η^*) in $\mathcal{U}_\gamma$. With Lemma IV.2 this substantial set could be reduced to any subset of $\mathcal{U}_\gamma$ that is not contained in a proper analytic variety of $\mathcal{U}_\gamma$.)

This can be expressed through the algebraic terminology used for Theorem IV.6. Condition (ii) of Theorem IV.1 is equivalent to $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)} \neq \emptyset$ for some $(\xi^*, \eta^*) \in \mathfrak{U}_\gamma$. Note $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)} \subsetneq \mathrm{HC}_\gamma$ unless $\mathrm{HC}_\gamma = \emptyset$; If V is in $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$ then $V + kW$,

$k \neq 0$ is in $\mathrm{HC}_\gamma \backslash \mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$, where W is any algebraic variety, excluding $w_0 = 0$, tangent to $\{w_2 = \xi^* w_0 + \eta^* w_1\}$ or containing the point $(0 : 1 : \eta^*)$. In fact it may occur that $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)} = \emptyset$ for certain (ξ^*, η^*) while HC_γ is not. (If HC_γ is non-empty, it is infinite dimensional.) As an example, consider γ to be the graph of the exponential map $z_2 = \exp(z_1)$ over the unit circle $\{z_1 \mid |z_1| = 1\}$. HC_γ consists of the holomorphic 1-chains that are representable as the graph of the exponential function over the unit disc $\{z_1 \mid |z_1| < 1\}$ plus a linear combination of algebraic varieties, excluding $w_0 = 0$. Due to tangency considerations, none of these holomorphic 1-chains are in $\mathrm{HC}_{\gamma,(1,1)}$. So $\mathrm{HC}_{\gamma,(1,1)}$ is empty. (In fact, by similar reasoning it can be shown that $\mathrm{HC}_{\gamma,((1-\tau)e^\tau, e^\tau)}$ is empty for $|\tau| < 1$.) As this example reveals, the info $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$ communicates about HC_γ may be nonexistent for some values of (ξ^*, η^*). Furthermore it is not apparent for which (ξ^*, η^*) this situation holds unless HC_γ has already been determined.

In light of this, we desire a condition equivalent to (i) that would only need testing about one (ξ^*, η^*) in $\mathcal{U}_\gamma$, preferably one of our own choosing. To achieve this we provide a modification (a weakening) of condition (ii) that isn't subject to the requirement that (ξ^*, η^*) be chosen outside of $\mathcal{T}_V$ and $\mathcal{I}_V$ for some V. In the algebraic terminology, we extend the isomorphism φ between $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)}$ and $\mathrm{SW}_{(\xi^*,\eta^*),G_\gamma}$ to an isomorphism between HC_γ and some broader class of "decompositions".

First we address the case of tangential intersections (which prohibited (ξ^*, η^*) from being chosen within $\mathcal{T}_V$). To handle this, we consider the elementary symmetric polynomials of f_j, instead of dealing with the f_j as individually well-defined functions. Accordingly, we translate the shockwave condition on each f_j into its equivalent condition on e_j, the elementary symmetric polynomials of f_j

Lemma VI.1. *Let $f_1, f_2, \ldots, f_N$ be continuous functions defined on some domain*

Ω. Let $e_1, e_2, \ldots$ be the elementary symmetric polynomials of these functions. (Note: $e_k = 0$ for $k > N$.) The following are equivalent.

1. *The functions $f_1, f_2, \ldots, f_N$ are analytic and satisfy the s.w. (shockwave) partial differential equation,*

$$f_j(f_j)_\xi = (f_j)_\eta, \text{ for } 1 \leq j \leq N \tag{6.1}$$

2. *The functions $e_1, e_2, \ldots, e_N$ are analytic and satisfy the e.s.p.s.w. (elementary symmetric polynomials of shockwave solutions) system of partial differential equations, defined as*

$$(e_1)_\xi e_k - (e_{k+1})_\xi = (e_k)_\eta, \text{ for } 1 \leq k \leq N, \text{(with } e_{N+1} := 0\text{)}. \tag{6.2}$$

Proof: Assume 1. Then the analyticity of the e_k is clear and 2 follows by the following calculation.

$$\begin{aligned}(e_1)_\xi e_k - (e_{k+1})_\xi &= \left(\sum_{\ell=1}^{N}(f_\ell)_\xi\right) \sum_{i_1<\cdots<i_k} f_{i_1}\cdots f_{i_k} - \left(\sum_{i_1<\cdots<i_{k+1}} f_{i_1}\cdots f_{i_{k+1}}\right)_\xi \\ &= \sum_{i_1<\cdots<i_k} \sum_{\ell=1}^{N}(f_\ell)_\xi f_{i_1}\cdots f_{i_k} - \sum_{i_1<\cdots<i_k} \sum_{\ell\notin\{i_1,\ldots,i_k\}} (f_\ell)_\xi f_{i_1}\cdots f_{i_k} \\ &= \sum_{i_1<\cdots<i_k} \sum_{\ell\in\{i_1,\ldots,i_k\}} (f_\ell)_\eta f_{i_1}\cdots \hat{f}_\ell \cdots f_{i_k} = (e_k)_\eta\end{aligned}$$

Now assume 2. The f_j are solutions to the polynomial with coefficients holomorphic in (ξ, η) given by

$$\zeta^N - e_1\zeta^{N-1} + \cdots + (-1)^N e_N = 0 \tag{6.3}$$

Analyticity of the f_j (off of a set, with codimension 1 in Ω, given as the union of vanishing sets of the discriminants of the irreducible factors of the above) follows from an argument similar to that presented in [2], Chapter 8, Section 2, which gives that solutions to polynomial equations with polynomial coefficients give locally holomorphic functions. As the f_j are a priori single-valued and continuous and by a removable singularities argument, the f_j are holomorphic on Ω.

Let (ξ^*, η^*) be any point in Ω. Let $\tilde{f}_j = f_j|_{\eta=\eta^*}$ and $\tilde{e_k} = e_k|_{\eta=\eta^*}$, be analytic functions in ξ on $\tilde{\Omega} = \Omega \cap \{\eta = \eta^*\}$. We point out that the Cauchy-Kovalevski theorem applies to both the s.w. equation and the e.s.p.s.w. system of equations with analytic Cauchy data along $\tilde{\Omega}$. (For our present purpose the form of the Cauchy-Kovalevski theorem given in [12], pg.16 is particularly well-suited.) Using the Cauchy-Kovalevski Theorem, we define F_j as the unique analytic function satisfying the s.w. differential equation on some neighborhood Ω' of (ξ^*, η^*) in Ω with the initial condition that F_j and $\tilde{f}_j$ agree on $\tilde{\Omega} \cap \Omega'$. Define E_k as the elementary symmetric functions of F_j. Note that E_k and $\tilde{e_k}$ agree on $\tilde{\Omega} \cap \Omega'$ and that by the previous implication E_k also satisfies the e.s.p.s.w. system of differential equations. Using the Cauchy-Kovalevski theorem, we see that E_j and e_j agree on Ω'. By the equality of the elementary symmetric polynomials of $\{f_j\}$ and $\{F_j\}$, their analyticity on Ω', and the agreement of f_j and F_j on $\tilde{\Omega} \cap \Omega'$, we have that f_j equals F_j on Ω'. Thus each f_j satisfies the s.w. equation.

$\square$

Now we formulate a elementary symmetric polynomial version of condition (ii) in Theorem IV.1. This new condition does not require that the perspective line be chosen to locally transversally intersect some holomorphic 1-chain bounded by γ. This allows more freedom on the choice of (ξ^*, η^*).

Theorem VI.2. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \mathbb{CP}^2$, the following* *are equivalent:*

(i) γ *bounds a holomorphic 1-chain within* $\mathbb{CP}^2$

(iii) $\exists\ (\xi^*, \eta^*)$ *with some neighborhood* Ω *such that* $\exists$ *non-negative integers* N^+ *and* N^- *and functions* $e_k^+(\xi, \eta)$ *for* $1 \le k \le N^+$ *and* $e_k^-(\xi, \eta)$ *for* $1 \le k \le N^-$ *that are defined on* Ω*, analytic in* (ξ, η)*, and satisfy the e.s.p.s.w. system of differential equations,* $(e_{k+1})_\xi + (e_k)_\eta = (e_1)_\xi e_k, \forall k \ge 1$*,* $(e = e_k^\pm)$*, (treating* $e^\pm_{N^\pm+1} = 0$*) such that on* Ω*,*

$$\frac{\partial^2}{\partial \xi^2} G_\gamma(\xi, \eta) = \frac{\partial^2}{\partial \xi^2}\left(e_1^+(\xi, \eta) - e_1^-(\xi, \eta)\right) \tag{6.4}$$

(iii') $\exists\ \eta^*$ *with neighborhood* Ω_η *such that any* (ξ^*, η^*) *with any connected neighborhood* $\Omega \subseteq \mathcal{U}_\gamma \cap (\mathbb{C} \times \Omega_\eta)$ *satisfies (iii).*

Before demonstrating the proof we establish the following lemma, an analog of Lemma IV.2. Recall the definitions for $\mathcal{U}_\gamma$, $\mathcal{T}_V$, and $\mathcal{I}_V$ given for that lemma.

Lemma VI.3. *Let* V *be a holomorphic 1-chain bounded by* γ *within* $\mathbb{CP}^2$ *and containing no components in the line at infinity,* $w_0 = 0$*. For* Ω *any component of* $\mathcal{U}_\gamma \backslash \mathcal{I}_V$*, there exist nonnegative integers* N^+ *and* N^- *and two sets of functions* $e_1^+, e_2^+, \ldots, e_{N^+}^+$ *and* $e_1^-, e_2^-, \ldots, e_{N^-}^-$ *well-defined, analytic in* (ξ, η)*, and satisfying the e.s.p.s.w. equations (that is (6.2)) on* Ω*, for which (6.4) holds.*

Proof (of Lemma): On Ω define $e_k^+(\xi, \eta)$ (resp. $e_k^-(\xi, \eta)$) to be the kth elementary symmetric function of the z_1 coordinates of positive (resp. negative) intersections, counting multiplicities, of the holomorphic chain V with line $w_2 = \xi w_0 + \eta w_1$. Also define $N^+(\xi, \eta)$ (resp. $N^-(\xi, \eta)$) to be the degree of such intersection. N^+ and N^- are locally constant and hence constant over Ω. The $e_k^\pm$ are guaranteed to be well-defined and continuous functions on Ω since $\Omega \subseteq \mathcal{U}_\gamma \backslash \mathcal{T}_V$. We understand by locally applying Lemma IV.2 and Lemma VI.1 that the $e_k^\pm$ are analytic, satisfy the e.s.p.s.w. system of equations and satisfy (6.4) on $\Omega \backslash \mathcal{T}_V$. Lemma 3 of [27] implies these properties extend to over all Ω.

$\square$

Proof (of Theorem): Note (iii') $\implies$ (iii) is clear and (i) $\implies$ (iii') follows from Lemma VI.3 and the observation that for $(\xi, \eta) \in \mathcal{U}_\gamma$ to lie in $\mathcal{T}_V$ is a property independent of ξ. It remains to show (iii) $\implies$ (i). With Theorem IV.1 it will suffice to show (iii) $\implies$ (ii).

Let (ξ^*, η^*), Ω, N^+, N^-, $e_1^+, \ldots, e_{N^+}^+$, and $e_1^-, \ldots, e_{N^-}^-$ be chosen in satisfying (iii). Define $\mathcal{O}(\Omega)$ as the ring of holomorphic functions on Ω. Define the monic polynomials Q^+ and Q^- in $\mathcal{O}(\Omega)[z]$ by the equation $Q(z) = z^N - e_1 z^{N-1} + e_2 z^{N-2} - \cdots + (-1)^N e_N$, where $Q = Q^\pm$, $N = N^\pm$, and $e_k = e_k^\pm$. Factor Q into irreducible factors $Q_1, Q_2, \ldots, Q_s$ and define D_t to be the discriminant of Q_t. Each D_t is an analytic function in (ξ, η) and vanishes wherever its corresponding factor Q_t does not have distinct roots. Since each Q_t is irreducible, no D_t is identically zero. By shrinking the neighborhood Ω and selecting a new (ξ^*, η^*), if necessary, we can assume to have non-vanishing discriminants for all irreducible factors of Q^+ and Q^- and that Ω is simply connected. Then we can define single-valued analytic functions $f_j^\pm$ on Ω as the roots of $Q^\pm$ counting multiplicity. Note the kth elementary symmetric polynomial of the functions f_j will be e_k. By Lemma VI.1, we then conclude each f_j satisfies the s.w. equation. To conclude simply note that (6.4) implies (4.2).

$\square$

Second we address the case of intersections between holomorphic 1-chains bounded by γ and the perspective line occurring at the line at ∞. To handle this situation we express the elementary symmetric polynomials in a homogenized form. In place of $e_1, e_2, \ldots, e_N$, we use $P_0, P_1, P_2, \ldots, P_N$, where $\frac{P_k}{P_0} = e_k$. Now we translate the e.s.p.s.w. system of equations into one expressed in this new form.

Lemma VI.4. *Let $e_1, e_2, \ldots, e_N$ ($e_{N+1} = 0$) be functions defined on some domain Ω. Let $P_0, P_1, P_2, \ldots, P_N$ ($P_{N+1} = 0$) be functions defined by $P_k = e_k P_0$, where P_0 is not identically zero. The following are equivalent.*

1. *The functions $e_1, e_2, \ldots, e_N$ and P_0 are analytic and satisfy the e.s.p.s.w. system of differential equations,*

$$(e_1)_\xi e_k - (e_{k+1})_\xi = (e_k)_\eta, \text{ for } 1 \le k \le N. \tag{6.5}$$

2. *The functions $P_0, P_1, \ldots, P_N$ are analytic, P_0 divides each function P_j within $\mathcal{O}(\Omega)$, and $P_0, P_1, \ldots, P_N$ satisfy the h.e.s.p.s.w. (homogenized e.s.p.s.w.) system of differential equations, defined as*

$$\begin{aligned} P_0[(P_{k+1})_\xi P_0 - P_{k+1}(P_0)_\xi + (P_k)_\eta P_0] \\ = P_k[(P_1)_\xi P_0 - P_1(P_0)_\xi + (P_0)_\eta P_0], \text{ for } k \ge 1. \end{aligned} \tag{6.6}$$

Proof: It is straightforward to see that the analyticity of $e_1, e_2, \ldots e_N$ and P_0 is equivalent to the analyticity of $P_0, P_1, \ldots, P_N$ and P_0 dividing each P_j within $\mathcal{O}(\Omega)$.

Completion of this proof results from the following basic calculation.

$$\left(\frac{P_1}{P_0}\right)_\xi \frac{P_k}{P_0} - \left(\frac{P_{k+1}}{P_0}\right)_\xi - \left(\frac{P_k}{P_0}\right)_\eta$$
$$= \frac{((P_1)_\xi P_0 - P_1(P_0)_\xi)P_k - ((P_{k+1})_\xi P_0 - P_{k+1}(P_0)_\xi)P_0 - ((P_k)_\eta P_0 - P_k(P_0)_\eta)P_0}{P_0^3}$$
$$= \frac{P_k[(P_1)_\xi P_0 - P_1(P_0)_\xi + (P_0)_\eta P_0] - P_0[(P_{k+1})_\xi P_0 - P_{k+1}(P_0)_\xi + (P_k)_\eta P_0]}{P_0^3}$$

□

Now assume $P_0, P_1, \ldots, P_N$ satisfy (6.6), not necessarily being derived from a set of $e_1, e_2, \ldots, e_N$ as in the previous lemma. We use $[P_0 : P_1 : \cdots : P_N]$ to denote the equivalence class from the relation $(P_0, P_1, \ldots, P_N) \sim (\lambda P_0, \lambda P_1, \ldots, \lambda P_N)$, for λ a meromorphic function not equivalently zero. If $P_0, P_1, \ldots, P_N$ satisfy the h.e.s.p.s.w. equations, then a basic calculation using (6.6) shows that $\lambda P_0, \lambda P_1, \ldots, \lambda P_N$ also satisfy the h.e.s.p.s.w. equations. So it is a well-defined notion to say that $[P_0 : P_1 : \cdots : P_N]$ satisfies the h.e.s.p.s.w. equations. While a $[P_0 : P_1 : \cdots : P_N]$ satisfying the h.e.s.p.s.w. equations has several representations, it is clear that we can choose $P_0, P_1, \ldots, P_N$ to be analytic. There are more specific classes of representations, which we now demonstrate.

Lemma VI.5. *Let $Q_0, Q_1, \ldots, Q_N$ be analytic functions on some domain Ω, with Q_0 not identically zero, that satisfy the h.e.s.p.s.w. equations. Then there exists $P_0, P_1, \ldots, P_N$ analytic on Ω such that $[Q_0 : Q_1 : \cdots : Q_N] = [P_0 : P_1 : \cdots : P_N]$, $(P_0)_\xi = 0$, and P_0 divides $((P_1)_\xi + (P_0)_\eta)$.*

Proof: This argument primarily uses the algebraic property of the unique factorization property of $\mathcal{O}(\Omega)$. Let $[P_0 : P_1 : \cdots : P_N]$ be a lowest terms representation of $[Q_0 : Q_1 : \cdots : Q_N]$ and assume for sake of contradiction that no lowest terms representation exists with $(P_0)_\xi = 0$. Then there exists an irreducible r that divides P_0 and $(ur)_\xi \not\equiv 0$ for all invertible u. Let $n > 0$ be the maximum number such that $r^n | P_0$. Since $[P_0 : P_1 : \cdots : P_N]$ is a lowest terms representation, there exists a $k \geq 1$ such that $r \not| P_k$. Let k be the smallest such that this is true. Now observe

$$P_0[(P_{k+1})_\xi P_0 - P_{k+1}(P_0)_\xi + (P_k)_\eta P_0] = P_k[(P_1)_\xi P_0 - P_1(P_0)_\xi + (P_0)_\eta P_0] \tag{6.7}$$

and that r^{2n-1} divides the left hand side and thus r^{2n-1} divides $\alpha := P_0(P_1)_\xi - (P_0)_\xi P_1 + P_0(P_0)_\eta$. Next note

$$P_{k-1}\alpha = (P_k)_\xi P_0^2 - P_k(P_0)_\xi P_0 + (P_{k-1})_\eta P_0^2 \tag{6.8}$$

(by the h.e.s.p.s.w if $k > 1$, and tautologically if $k = 1$). r^{2n} divides the left-hand side and the first and third terms on the right-hand side, so $r^{2n} | P_k(P_0)_\xi P_0$. This implies that $r^n | (P_0)_\xi$. By applying the product rule to a factorization of P_0, we see that $r | r_\xi$. This in turn implies r is the product of an invertible element and an element that is constant with respect to ξ ($r_\xi = kr \implies r = A(\eta)\exp(\int k \, d\xi)$). This achieves the desired contradiction.

So let $[P_0 : P_1 : \cdots : P_N]$ be a lowest terms representation of $[Q_0 : Q_1 : \cdots : Q_N]$ such that $(P_0)_\xi = 0$. By the h.e.s.p.s.w. equations this implies

$$P_0[(P_{k+1})_\xi + (P_k)_\eta] = P_k[(P_1)_\xi + (P_0)_\eta], \text{ for } 1 \leq k \leq N, (\text{with } P_{N+1} := 0). \tag{6.9}$$

Now let r be any irreducible factor that divides P_0 with multiplicity $m > 0$. By the lowest-term representation, there exists a $k \geq 1$ such that $r \not| P_k$. Thus by (6.9), $r^m | ((P_1)_\xi + (P_0)_\eta)$. As this holds for all factors of P_0, it follows that P_0 divides $(P_1)_\xi + (P_0)_\eta$.

$\square$

Let $P_0, P_1, \ldots, P_N$ be analytic functions such that P_0 doesn't identically vanish.

We call $P_0, P_1, \ldots, P_N$ a *refined representative* of $[P_0 : P_1 : \cdots : P_N]$, if it satisfies (6.9) and $(P_0)_\xi = 0$. We call the combination of (6.9) and $(P_0)_\xi = 0$, the r.h.e.s.p.s.w. system of differential equations.

We call $P_0, P_1, \ldots, P_N$ a *special representative* of $[P_0 : P_1 : \cdots : P_N]$, if it satisfies the statement of Lemma VI.5. We say that $P_0, P_1, \ldots, P_N$ satisfy the s.h.e.s.p.s.w. system of equations on Ω if they satisfy one of the following equivalent conditions.

1. Equation (6.9) and $(P_0)_\xi = 0$ hold and $P_0, P_1, \ldots, P_N$ have no common irreducible factors in $\mathcal{O}(\Omega)$.
2. Equation (6.9), $(P_0)_\xi = 0$, and $P_0|((P_1)_\xi + (P_0)_\eta)$ all hold.
3. There exists an analytic function μ such that

$$(P_{k+1})_\xi + (P_k)_\eta = \mu P_k, \text{ for } 0 \leq k \leq N, (\text{with } P_{N+1} := 0), \tag{6.10}$$

   and $(P_0)_\xi = 0$.

4. There exists an analytic function μ such that

$$(P_{k+1})_\xi + (P_k)_\eta = \mu P_k, \text{ for } -1 \leq k \leq N, (\text{with } P_{-1} := 0,\ P_{N+1} := 0) \tag{6.11}$$

Proof of Equivalence: The proof of Lemma VI.5 immediately gives that $1 \implies 2$. If we assume 2, then the proof of Lemma VI.5 and the previous implication give that there exist analytic functions $R_0, R_1, \ldots, R_N$ which satisfy 1, 2, and that $[P_0 : P_1 : \cdots : P_N] = [R_0 : R_1 : \cdots : R_N]$. (Note that it was assumed that P_0 was not equivalently zero, thus this holds for R_0 as well.) As $R_0, R_1, \ldots, R_N$ have no common irreducible factors in $\mathcal{O}(\Omega)$, there exists a $\lambda \in \mathcal{O}(\Omega)$ such that $P_i = \lambda R_i$, for all i. By noting $(R_0)_\xi = 0$ and $(P_0)_\xi = 0$ it holds that $\lambda_\xi = 0$. Next $R_0|((R_1)_\xi + (R_0)_\eta)$

and $P_0|((P_1)_\xi + (P_0)_\eta)$ implies $\lambda|\lambda_\eta$. Together $\lambda_\xi = 0$ and $\lambda|\lambda_\eta$ imply that λ is non-vanishing and thus a unit in $\mathcal{O}(\Omega)$. Thus $P_0, P_1, \ldots, P_N$ has no common irreducible factors and 1 holds.

Assuming 2, let $\mu = \frac{(P_1)_\xi + (P_0)_\eta}{P_0}$, which is an analytic function. Then note (6.10) holds for $k = 0$ tautologically, and it holds for other k by dividing equation (6.9) through by P_0. So 3 holds.

Assuming 3, note by (6.10) for $k = 0$ that $(P_1)_\xi + (P_0)_\eta = \mu P_0$. This implies that $P_0|((P_1)_\xi + (P_0)_\eta)$ and (6.9) by cross-multiplication of this equation with (6.10) for $k \geq 1$. So 2 holds.

3 and 4 are equivalent by recognizing that $(P_0)_\xi = 0$ is equivalent to (6.11) for $k = -1$ considering $P_{-1} = 0$.

$\square$

If $P_0, P_1, \ldots, P_N$ is a special representative, then the other special representatives are of the form $[\lambda P_0 : \lambda P_1 : \cdots : \lambda P_N]$ where $\lambda_\xi = 0$ and λ is invertible (i.e. non-vanishing on Ω).

Now an easy calculation shows that if $[P_0 : P_1 : \cdots : P_N]$ satisfy the s.h.e.s.p.s.w. equations for a particular μ, then for λ non-vanishing on Ω with $\lambda_\xi = 0$, it holds that $[\hat{P}_0 : \hat{P}_1 : \cdots : \hat{P}_N]$, with $\hat{P}_k = \lambda P_k$, satisfies the s.h.e.s.p.s.w. equations with $\hat{\mu} = \mu + \frac{\lambda_\eta}{\lambda}$. So μ_ξ remains invariant while $\mu|_{\xi=\xi^*}$ does not. (In fact $\mu_\xi = \left(\frac{P_1}{P_0}\right)_{\xi\xi}$.)

If we can define $\lambda = \exp(-\int_{\eta^*}^{\eta} \mu(\xi, \eta')|_{\xi=\xi^*}\, d\eta')$, then $\hat{\mu}|_{\xi=\xi^*} = 0$. But to generally define λ as such requires some properties on Ω. λ will be well-defined, if $(\xi_0, \eta_0) \in \Omega$ implies that $\{\eta \mid (\xi_0, \eta) \in \Omega\}$ is connected, is simply-connected, and contains η^*. For instance we could assume that Ω is a polydisc (with respect to the coordinate directions) or that it is a complete Reinhardt domain centered at (ξ^*, η^*). Either of these would be appropriately suited. As our interest truly lies locally, the choice of type of neighborhood is less material. We use polydiscs in our treatment.

If $P_0, P_1, \ldots, P_N$ satisfy the s.h.e.s.p.s.w. equations with μ such that $\mu_{\xi=\xi^*} = 0$, then we say that it is a *canonical representative* of $[P_0 : P_1 : \cdots : P_N]$ and that it satisfies the c.h.e.s.p.s.w. equations. Note two canonical representatives of $[P_0 : P_1 : \cdots : P_N]$ will differ only by a scalar (complex) multiple.

In conclusion we derive the following theorem.

Theorem VI.6. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \mathbb{CP}^2$, the following are equivalent:*

(i) γ bounds a holomorphic 1-chain within $\mathbb{CP}^2$

(iv) $\exists$ (ξ^*, η^*) with some neighborhood Ω such that $\exists$ non-negative integers N^+ and N^- and functions $P_k^+(\xi,\eta)$ for $0 \le k \le N^+$ and $P_k^-(\xi,\eta)$ for $0 \le k \le N^-$, with $P_0^\pm \not\equiv 0$, that are defined on Ω, analytic in (ξ,η), and satisfy the s.h.e.s.p.s.w. system of differential equations, $(P_{k+1})_\xi + (P_k)_\eta = \mu P_k, \forall k \ge 0$, $(P_0)_\xi = 0$, for some analytic function μ, ($\mu = \mu^\pm$ and $P = P_k^\pm$) (treating $P^\pm_{N^\pm+1} = 0$), such that on Ω,

$$\frac{\partial^2}{\partial \xi^2} G_\gamma(\xi,\eta) = \frac{\partial^2}{\partial \xi^2}\left(\frac{P_1^+(\xi,\eta)}{P_0^+(\xi,\eta)} - \frac{P_1^-(\xi,\eta)}{P_0^-(\xi,\eta)}\right) \tag{6.12}$$

(iv') Any (ξ^*, η^*) with any connected neighborhood $\Omega \subseteq \mathfrak{U}_\gamma$ satisfies (iv).

(iv") Any (ξ^*, η^*) with any polydisc neighborhood (with respect to the coordinates (ξ,η)) $\Omega \subseteq \mathfrak{U}_\gamma$ satisfies (iv) with the additional restriction that $\mu(\xi,\eta)|_{\xi=\xi^*}$ is identically zero as a function in η.

Before giving the proof of this theorem we establish the following lemma, another analog of Lemma IV.2.

Lemma VI.7. *Let V be a holomorphic 1-chain bounded by γ containing no components in the line at infinity, $\{w_0 = 0\}$. For Ω any component of $\mathcal{U}_\gamma$, there exist non-negative integers N^+ and N^- and functions $P_0^+, P_1^+, \ldots, P_{N^+}^+$ and $P_0^-, P_1^-, \ldots, P_{N^-}^-$ that are well-defined, analytic in (ξ, η), and satisfy the s.h.e.s.p.s.w. equations on Ω, for which (6.12) holds.*

Proof (of Lemma): Let's assume that V is a positive holomorphic 1-chain. If we establish the lemma in this case with $N^- = 0$, then the lemma holds in general as any holomorphic 1-chain is the difference of two positive holomorphic 1-chains.

Let $\mathcal{E} = \{(\xi, \eta) \mid V$ has a component contained in the line $\{w_2 = \xi w_0 + \eta w_1\}\}$. Note that this set is finite. Choose a point $(\xi^*, \eta^*) \in \Omega \cap \mathcal{I}_V \backslash \mathcal{E}$. Define $q = (0 : 1 : \eta^*)$ and note that q is in the intersection of V and $\{w_0 = 0\}$. Let m denote the multiplicity of intersection between V and $\{w_0 = 0\}$ at q.

Note $\frac{w_2 - \xi^* w_0 - \eta^* w_1}{w_1}$ and $\frac{w_0}{w_1}$ serve as natural holomorphic coordinates near q. We'll use x to denote $\frac{w_2 - \xi^* w_0 - \eta^* w_1}{w_1}$ and y to denote $\frac{w_0}{w_1}$. Let U be a polydisc in the holomorphic coordinates x and y. Specifically let U be $\{|x| < \delta, |y| < \epsilon\}$ for an appropriate choice of δ and ϵ to be given. Choose ϵ so that V does not intersect $\{x = 0, |y| = \epsilon\}$ (which can be done since (ξ^*, η^*) is outside $\mathcal{E}$). Choose δ small enough such that V does not intersect $\{|x| \leq \delta, |y| = \epsilon\}$ and so that V only intersects $y = 0$ inside U at q. Also make sure δ and ϵ are chosen small enough that V can be defined in U as the divisor of a function F holomorphic in x and y on U. It should be noted that F is divisible by neither x nor y.

Define $\Omega_0 = \Omega \cap \{(\xi, \eta) \mid \epsilon|\xi - \xi^*| + |\eta - \eta^*| < \delta\}$, shrinking δ if necessary to ensure that Ω_0 is connected. For (ξ, η) in Ω_0 the line $\{w_2 = \xi w_0 + \eta w_1\}$ (or $\{x = (\xi - \xi^*)y + (\eta - \eta^*)\}$) only intersects ∂U at $\{|x| < \delta, |y| = \epsilon\}$. Let $H_{\xi,\eta}(y) =$

$F((\xi-\xi^*)y+(\eta-\eta^*),y)$ and note it is non-vanishing on $|y|=\epsilon$ for $(\xi,\eta)\in\Omega_0$. Thus

$$\frac{1}{2\pi \mathrm{i}}\int_{|y|=\epsilon}\frac{H'_{\xi,\eta}(y)}{H_{\xi,\eta}(y)}\,dy \tag{6.13}$$

is well-defined, integer-valued, and analytic with respect to (ξ,η) in Ω_0. Hence the degree of intersection between V and the line $\{w_2=\xi w_0+\eta w_1\}$ inside U is constant for (ξ,η) in Ω_0.

Then we can define the following functions analytically on Ω_0. Let $e_{0,k}(\xi,\eta)$ be the kth elementary symmetric function of the z_1 coordinates of the intersections (counting multiplicity) between V and $\{w_2=\xi w_0+\eta w_1\}$ inside U. Now our aim is prove the following estimate on $\Omega_0\backslash\mathfrak{I}_V$.

$$e_{0,k}(\xi,\eta)\le\frac{C}{|\eta-\eta^*|^m},\qquad\text{for all } k,\qquad\text{for some constant } C \tag{6.14}$$

We'll also demonstrate that there is always a k for which this estimate is sharp, in that there is always a k for m cannot be replaced with something strictly smaller.

Now define $c_{0,k}(\xi,\eta)$ to be the sum of kth powers of the z_1 (or $1/y$) coordinates of intersection inside U between V and $\{w_2=\xi w_0+\eta w_1\}$. The following equality arises from basic residue calculations.

$$c_{0,k}(\xi,\eta)=\frac{1}{2\pi \mathrm{i}}\int_{|y|=\epsilon}\frac{1}{y^k}\frac{H'_{\xi,\eta}(y)}{H_{\xi,\eta}(y)}\,dy-\frac{1}{(k-1)!}\frac{d^{k-1}}{dy^{k-1}}\left(\frac{H'_{\xi,\eta}(y)}{H_{\xi,\eta}(y)}\right)\Bigg|_{y=0} \tag{6.15}$$

For simplicity we'll define

$$S_k=\frac{1}{2\pi \mathrm{i}}\int_{|y|=\epsilon}\frac{1}{y^k}\frac{H'_{\xi,\eta}(y)}{H_{\xi,\eta}(y)}\,dy \tag{6.16}$$

Note that the S_k are bounded functions in ξ and η.

Elementary symmetric functions can be given in terms of sums of powers, so we can relate $e_{0,k}$ in terms of $c_{0,k}$. This is most easily calculated through generating functions. Let $E(t) = \sum_{k=0}^{\infty} e_{0,k} t^k$ and $C(t) = \sum_{k=1}^{\infty} c_{0,k} t^{k-1}$. Now using (6.15) we get that

$$
\begin{aligned}
\text{(6.17)} \quad C(t) &= \sum_{k=1}^{\infty} S_k t^{k-1} - \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial y^{k-1}} \left(\frac{H'_{\xi,\eta}(y)}{H_{\xi,\eta}(y)} \right) \bigg|_{y=0} t^{k-1} \\
&= \sum_{k=1}^{\infty} S_k t^{k-1} - \frac{H'_{\xi,\eta}(t)}{H_{\xi,\eta}(t)}
\end{aligned}
$$

It's a result of symmetric function theory that $C(-t) = \frac{E'(t)}{E(t)}$. (For instance see [20].) Thus $E(t) = \exp\left(\int_0^t C(-\tau)\, d\tau\right)$. Using (6.17) we get

$$
\begin{aligned}
\text{(6.18)} \quad E(t) &= \exp\left(\sum_{k=1}^{\infty} \frac{-S_k}{k} (-t)^k \right) \frac{H_{\xi,\eta}(-t)}{H_{\xi,\eta}(0)} \\
&= \exp\left(\sum_{k=1}^{\infty} \frac{-S_k}{k} (-t)^k \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{H^{(k)}_{\xi,\eta}(0)}{H_{\xi,\eta}(0)} t^k
\end{aligned}
$$

Examining the above, we can see that $e_{0,k}$ is a linear combination of $\frac{H^{(j)}_{\xi,\eta}(0)}{H_{\xi,\eta}(0)}$ for $j \leq k$ with coefficients being expressions in terms of the S_k. By calculating the multiplicity of intersection between $y = 0$ and $F(x, y) = 0$ at q, we obtain that $F(x, 0)$ has order m in x. This implies that $H_{\xi,\eta}(0)$ (which equals $F(\eta - \eta^*, 0)$) can be bounded below by a constant multiple of $(\eta - \eta^*)^m$. Since $H^{(j)}_{\xi,\eta}(0)$ is a bounded function in ξ and η this yields the desired estimate (6.14).

Choose the smallest k such that $H^{(k)}_{\xi,\eta}(0)$ doesn't vanish at $\eta = \eta^*$. (This is equivalent to the smallest k such that $\frac{\partial^k}{\partial y^k} F(x, y)\Big|_{(x,y)=(0,0)} \neq 0$. There must be such a k as F is not divisible by x.) Then $\frac{H^{(j)}_{\xi,\eta}(0)}{H_{\xi,\eta}(0)}$ is comparable to $\frac{1}{|\eta - \eta^*|^m}$ when j equals k and for no lesser value of j. Thus $|e_{0,k}|$ will be comparable to $\frac{1}{|\eta - \eta^*|^m}$, and so we see there will always exist a k for which (6.14) is sharp.

Now let $\{q_s\}$ represent all the points of intersection between V and $\{w_0 = 0\}$, given with multiplicity of intersection m_s. Let

$$
\text{(6.19)} \qquad P_0^+(\xi, \eta) = \prod_{s \mid q_s \neq (0:0:1)} \left(\eta - (z_2/z_1)\big|_{q_s}\right)^{m_s}
$$

By Lemma VI.3 we have N^+ determined and, for $1 \leq k \leq N^+$, $e_k^+(\xi,\eta)$ defined on $\Omega\backslash\mathfrak{I}_V$ which are analytic and satisfy the e.s.p.s.w. equations as well as satisfying (6.4). By Lemma VI.4 we have that $P_k^+ = P_0^+ e_k^+$ satisfy the h.e.s.p.s.w. equations. We also see that (6.12) holds on this domain. Using (6.14) we can see the P_k^+ are locally bounded near $\Omega \cap \mathfrak{I}_V$ and so by Lemma 3 of [27] we have that the functions P_k^+ analytically extend to Ω and so satisfy the h.e.s.p.s.w. equations and (6.12) on Ω. The sharpness of the estimate (6.14) in fact implies that $[P_0^+ : P_1^+ : \cdots : P_{N^+}^+]$ is in fact in lowest terms. This and the fact that $(P_0^+)_\xi \equiv 0$ implies through the proof of Lemma VI.5 the P_k^+ satisfy the s.h.e.s.p.s.w. equations.

$\square$

Proof (of Theorem): First we point out the trivial implications, (iv") $\implies$ (iv). Also by the argument immediately preceding the statement of the theorem on the freedom of choice for μ we see that (iv') $\implies$ (iv"). By Lemma VI.7 we have that (i) $\implies$ (iv'). So it only remains to show that (iv) $\implies$ (i). By Theorem VI.2 it suffices to show that (iv) $\implies$ (iii).

Let (ξ^*,η^*), N^+, N^-, $P_0^+,\ldots,P_{N^+}^+$, $P_0^-,\ldots,P_{N^-}^-$, and μ be chosen according to satisfying (iv). Note since P_0^+ and P_0^- are not identically zero, we can shrink the neighborhood and choose a different (ξ^*,η^*), if necessary, such that $P_0^\pm$ are non-vanishing over the given neighborhood. Now define $e_k^\pm = \frac{P_k^\pm}{P_0^\pm}$, which are well-defined and analytic in the given neighborhood of (ξ^*,η^*). Note (where $e_k = e_k^\pm$ and $P_k = P_k^\pm$) that

$$
\begin{aligned}
(6.20)\quad (e_{k+1})_\xi + (e_k)_\eta &= \frac{[(P_{k+1})_\xi + (P_k)_\eta]P_0 - P_k(P_0)_\eta}{P_0^2} \\
&= \frac{(\mu P_0 - (P_0)_\eta)P_k}{P_0^2} = \frac{(P_1)_\xi}{P_0}\frac{P_k}{P_0} = (e_1)_\xi e_k.
\end{aligned}
$$

Then noting (6.12) implies (6.4), we see that (iii) is satisfied.

□

As Theorem VI.2 and Theorem VI.6 are analogs of Theorem IV.1, we may also produce analogs of Theorem IV.6. To do so will require construction of the appropriate algebraic objects.

Define $\mathrm{HC}_{\gamma,\eta^*}$ to be the affine subspace of HC_γ containing holomorphic 1-chains bounded by γ that do not intersect $\{w_2 = \eta^* w_1\}$ at the line at infinity, or equivalently, those that don't intersect the point $[0 : 1 : \eta^*]$. Note $\mathrm{HC}_{\gamma,(\xi^*,\eta^*)} \subset \mathrm{HC}_{\gamma,\eta^*} \subset \mathrm{HC}_\gamma$. Accordingly we wish to define spaces for the broader types of decompositions we've previously discussed to extend $\mathrm{SW}_{(\xi,\eta^*),\gamma}$. Also we wish to extend the isomorphism φ to an isomorphism on these extended spaces. Before plunging into the remaining algebraic construction, which are quite technical, we give the following diagram to illustrate our objective.

$$
(6.21)\qquad
\begin{array}{ccccc}
\mathrm{HC}_{\gamma,(\xi^*,\eta^*)} & \xrightarrow{\subset} & \mathrm{HC}_{\gamma,\eta^*} & \xrightarrow{\subset} & \mathrm{HC}_\gamma \\
\cong\big\downarrow\varphi & & \cong\big\downarrow\varphi^E & & \cong\big\downarrow\Phi \\
\mathrm{SW}_{(\xi^*,\eta^*),G_\gamma} & \xrightarrow{\subset} & \mathrm{ESPSW}_{(\xi^*,\eta^*),G_\gamma} & \xrightarrow{\subset} & \mathrm{HESPSW}_{(\xi^*,\eta^*),G_\gamma} \\
\big\downarrow\cap & & \big\downarrow\cap & & \big\downarrow\cap \\
\mathrm{SW}_{(\xi^*,\eta^*)} & \xrightarrow{\subset} & \mathrm{ESPSW}_{(\xi^*,\eta^*)} & \xrightarrow{\subset} & \mathrm{HESPSW}_{(\xi^*,\eta^*)}
\end{array}
$$

Recall $\mathcal{O}_{(\xi^*,\eta^*)}$ is the ring of germs of analytic function about (ξ^*,η^*). Define $\mathcal{M}_{(\xi^*,\eta^*)} = \mathsf{ff}(\mathcal{O}_{(\xi^*,\eta^*)})$ as the field of germs of meromorphic functions about (ξ^*,η^*).

Define $\mathrm{FS}_{(\xi^*,\eta^*)}$ (for the curious, FS stands for finite sequence) to be the monoid (semigroup with identity) $\mathcal{O}_{(\xi^*,\eta^*)}[\zeta]\backslash\{0\}$ with the monoid operation being polynomial multiplication, which we denote by $\circ$. An element $P(\zeta)$ of $\mathrm{FS}_{(\xi^*,\eta^*)}$ should be perceived as a finite sequence of germs (starting with the coefficient of the leading term, which we call P_0 to denote its precedence in the sequence). The ratios with each germ in the sequence with the non-vanishing leading term serve as elementary symmetric polynomials of some objects $\lambda_1, \lambda_2, \ldots, \lambda_N$, by considering $P(\zeta) = P_0 \prod_{j=1}^{N}(\zeta + \lambda_j)$. So an element of $\mathrm{FS}_{(\xi^*,\eta^*)}$ corresponds to a formal finite sum of these objects. From this point of view, the monoid operation of polynomial multiplication is essentially addition amongst nonnegative formal linear combinations of these objects, and so $\circ$ serves as our "linear" structure. (Note: This structure provides addition, not subtraction.)

In general we may think of the objects $\lambda_1, \lambda_2, \ldots, \lambda_N$ as multi-valued meromorphic germs about (ξ^*, η^*). (For our definition of multi-valued meromorphic germs in dimension 1, which we also refer to as Laurent-Puiseux germs, see the introduction to Appendix A.) If $P(\zeta)$ is a monic polynomial, then these objects are multi-valued holomorphic germs. If $P(\zeta)$ factors into linear factors, then these objects are single-valued meromorphic germs. (Of course if both these properties hold for $P(\zeta)$, then these objects are single-valued holomorphic germs.)

Now define the submonoid $\mathrm{RHESPSW}^+_{(\xi^*,\eta^*)}$ to be all elements $P_0\zeta^N + P_1\zeta^{N-1} + \cdots + P_N$ of $\mathrm{FS}_{(\xi^*,\eta^*)}$ such that $P_0, P_1, \ldots, P_N$ satisfy (6.9) and $(P_0)_\xi = 0$. (This is equivalent to $P_0, P_1, \ldots, P_N$ satisfying the r.h.e.s.p.s.w. system of differential equations.) The identity 1 clearly is an element, so to verify that this is a submonoid let's examine the sum (by $\circ$) of two general elements. Let $P = \sum_{i=0}^{M} P_i\zeta^{M-i}$ and $Q = \sum_{i=0}^{N} Q_i\zeta^{N-i}$ be two elements of $\mathrm{RHESPSW}^+_{(\xi^*,\eta^*)}$ and let $R = P \circ Q$. So

$R = \sum_{i=0}^{M+N} R_i \zeta^{M+N-i}$, where $R_i = \sum_{j=0}^{i} P_j Q_{i-j}$. R can be seen to be an element of $\mathrm{RHESPSW}^+_{(\xi^*,\eta^*)}$ by the following calculation.

$$
\begin{aligned}
(6.22)\quad & R_0 \left[(R_{k+1})_\xi + (R_k)_\eta\right] \\
&= P_0 Q_0 \left[\sum_{j=1}^{k+1} (P_j)_\xi Q_{k+1-j} + \sum_{j=0}^{k} P_j (Q_{k+1-j})_\xi + \sum_{j=0}^{k} (P_j)_\eta Q_{k-j} + \sum_{j=0}^{k} P_j (Q_{k-j})_\eta\right] \\
&= \sum_{j=0}^{k} P_0 \left[(P_{j+1})_\xi + (P_j)_\eta\right] Q_0 Q_{k-j} + \sum_{j=0}^{k} P_0 P_j Q_0 \left[(Q_{k-j+1})_\xi + (Q_{k-j})_\eta\right] \\
&= \sum_{j=0}^{k} P_j \left[(P_1)_\xi + (P_0)_\eta\right] Q_0 Q_{k-j} + \sum_{j=0}^{k} P_0 P_j Q_{k-j} \left[(Q_1)_\xi + (Q_0)_\eta\right] \\
&= \left(\sum_{j=0}^{k} P_j Q_{k-j}\right) \left[(P_1 Q_0 + P_0 Q_1)_\xi + (P_0 Q_0)_\eta\right] \\
&= R_k \left[(R_1)_\xi + (R_0)_\eta\right]
\end{aligned}
$$

Define $\mathrm{ESPSW}^+_{(\xi^*,\eta^*)}$ to be the submonoid of monic polynomials in $\mathrm{RHESPSW}^+_{(\xi^*,\eta^*)}$. Define $\mathrm{HESPSW}^+_{(\xi^*,\eta^*)}$ to be $(\mathrm{RHESPSW}^+_{(\xi^*,\eta^*)})/\sim$, where we say $P \sim Q$ if $Q = \lambda P$, for a nonzero λ in $\mathcal{M}_{(\xi^*,\eta^*)}$ (or effectively $\mathcal{M}_{\eta^*}$). Or equivalently define $\mathrm{HESPSW}^+_{(\xi^*,\eta^*)}$ as the quotient monoid $\mathrm{RHESPSW}^+_{(\xi^*,\eta^*)}/(\mathcal{O}_{(\xi^*,\eta^*)}\backslash\{0\})$. Define $\mathrm{SW}^+_{(\xi^*,\eta^*)}$ to be the submonoid of $\mathrm{SW}_{(\xi^*,\eta^*)}$ consisting of the nonnegative formal linear combinations

There is a natural inclusion of $\mathrm{SW}^+_{(\xi^*,\eta^*)}$ into $\mathrm{ESPSW}^+_{(\xi^*,\eta^*)}$ given by $\sum \mu_j f_j \mapsto \prod(\zeta + f_j)^{\mu_j}$. Also because $\mathrm{ESPSW}^+_{(\xi^*,\eta^*)} \cap (\mathcal{O}_{(\xi^*,\eta^*)}\backslash\{0\}) = \{1\}$, there is a natural inclusion map of $\mathrm{ESPSW}^+_{(\xi^*,\eta^*)}$ into $\mathrm{HESPSW}^+_{(\xi^*,\eta^*)}$ through $\mathrm{RHESPSW}^+_{(\xi^*,\eta^*)}$. Therefore $\mathrm{SW}^+_{(\xi^*,\eta^*)} \subset \mathrm{ESPSW}^+_{(\xi^*,\eta^*)} \subset \mathrm{HESPSW}^+_{(\xi^*,\eta^*)}$.

Here is one way of understanding these objects, building on the previously described intuition concerning $\mathrm{FS}_{(\xi^*,\eta^*)}$. $\mathrm{SW}^+_(\xi^*,\eta^*)$ is already well understood as a formal finite sum (meaning no negatively counted terms) of holomorphic germs that satisfy the shockwave equation. An element of $\mathrm{ESPSW}^+_{(\xi^*,\eta^*)}$ can be thought of as a formal finite sum of multi-valued holomorphic germs that locally satisfy the shockwave equation. And an element of $\mathrm{HESPSW}^+_{(\xi^*,\eta^*)}$ may be perceived as a formal finite sum of multi-valued meromorphic germs that locally satisfy the shockwave equation. The next step is to derive formal linear combinations from formal finite sums.

For a general commutative monoid M with the cancellation property, one can construct the group of fractions of M, which we'll denote as $K(M)$ (the K-functor as referenced in K-theory). ($K(M)$ can also be called the group of differences of M, which in fact bodes better with our thinking of $\circ$.) The elements of $K(M)$ may be represented as p/q, where p and q are elements of M. We identify the elements p/q and $(\lambda p)/(\lambda q)$ as being equal for any λ in M. (A notation suggesting this as the group of differences, we could represent the elements as $p-q$ identifying $(\lambda+p)-(\lambda+q)$.)

Now define $\mathrm{ESPSW}_{(\xi^*,\eta^*)}$ to be the group of fractions(differences) of $\mathrm{ESPSW}^+_{(\xi^*,\eta^*)}$ and $\mathrm{HESPSW}_{(\xi^*,\eta^*)}$ to be the group of fractions(differences) of $\mathrm{HESPSW}^+_{(\xi^*,\eta^*)}$, so that both now have a true group operation (which we consider as our $\mathbb{Z}$-linear structure) which was derived from the original monoid operation. Also note that $\mathrm{SW}_{(\xi^*,\eta^*)}$ is equivalent to the group of fractions(differences) of $\mathrm{SW}^+_{(\xi^*,\eta^*)}$. Applying the functor K preserves these inclusions, and we have $\mathrm{SW}_{(\xi^*,\eta^*)} \subset \mathrm{ESPSW}_{(\xi^*,\eta^*)}$, where $\sum f_j^+ - \sum f_j^-$ becomes identified as $\prod(\zeta + f_j^+)/(\prod(\zeta + f_j^-))$ and $\mathrm{ESPSW}_{(\xi^*,\eta^*)} \subset \mathrm{SHESPSW}_{(\xi^*,\eta^*)}$, where P/Q becomes represented as $[P]/[Q]$.

Now recall $\mathrm{SW}_{(\xi^*,\eta^*)}$ has a natural homomorphism into $\mathcal{O}_{(\xi^*,\eta^*)}$, simply given by evaluating the element $\sum f_j^+ - \sum f_j^-$ non-formally within $\mathcal{O}_{(\xi^*,\eta^*)}$. This homomorphism can be factored through the inclusion of $\mathrm{SW}_{(\xi^*,\eta^*)}$ into $\mathrm{ESPSW}_{(\xi^*,\eta^*)}$. For P/Q in $\mathrm{ESPSW}_{(\xi^*,\eta^*)}$, the corresponding homomorphism is given as $P/Q \mapsto P_1 - Q_1$. For $\mathrm{HESPSW}_{(\xi^*,\eta^*)}$ the natural homomorphism must instead be given into $\mathcal{M}_{(\xi^*,\eta^*)}$ and is defined by $[P]/[Q] \mapsto \frac{P_1}{P_0} - \frac{Q_1}{Q_0}$.

For $G \in \mathcal{O}_{(\xi^*,\eta^*)}$, $\mathrm{SW}_{(\xi^*,\eta^*)}$ is the inverse image under this homomorphism of the functions agreeing with G in the second ξ derivative. Summarily we make the analogous definitions for $\mathrm{ESPSW}_{(\xi^*,\eta^*),G}$ and $\mathrm{HESPSW}_{(\xi^*,\eta^*),G}$. As a result we have the inclusion $\mathrm{SW}_{(\xi^*,\eta^*),G} \subset \mathrm{ESPSW}_{(\xi^*,\eta^*),G} \subset \mathrm{HESPSW}_{(\xi^*,\eta^*),G}$.

We define φ^E as the homomorphism from $\mathrm{HC}_{\gamma,\eta^*}$ to $\mathrm{ESPSW}_{(\xi^*,\eta^*),G_\gamma}$ given by the procedure of Lemma VI.3. And similarly we define Φ as the homomorphism from HC_γ to $\mathrm{HESPSW}_{(\xi^*,\eta^*),G_\gamma}$ given by the procedure of Lemma VI.7.

In fact these two homomorphisms are isomorphism. We state this in the final two theorems of this section.

Theorem VI.8. *The affine spaces* $\mathrm{HC}_{\gamma,\eta^*}$ *and* $\mathrm{ESPSW}_{(\xi^*,\eta^*),G_\gamma}$ *are isomorphic via the map* φ^E*.*

Proof: For a generic choice of ξ close to ξ^*, elements of $\mathrm{HC}_{\gamma,\eta^*}$ may be viewed as elements of $\mathrm{HC}_{\gamma,(\xi,\eta^*)}$ and elements of $\mathrm{ESPSW}_{(\xi^*,\eta^*),G_\gamma}$ may be viewed as elements of $\mathrm{SW}_{(\xi,\eta^*),G_\gamma}$. In fact elements of $\mathrm{HC}_{\gamma,\eta^*}$ and $\mathrm{ESPSW}_{(\xi^*,\eta^*),G_\gamma}$ may be uniquely defined by these elements of "perturbate base". The homomorphism φ^E corresponds to φ under this type of perturbation operation, so it suffices to note Theorem IV.6.

$\square$

Theorem VI.9. *The affine spaces* HC_γ *and* $\mathrm{HESPSW}_{(\xi^*,\eta^*),G_\gamma}$ *are isomorphic via the map* Φ*.*

Proof: (What follows is a word-processor proof, bearing the same argument structure as the proof of the previous theorem.) For a generic choice of η close to η^*, elements of HC_γ may be viewed as elements of $\mathrm{HC}_{\gamma,\eta}$ and elements of $\mathrm{HESPSW}_{(\xi^*,\eta^*),G_\gamma}$ may be viewed as elements of $\mathrm{ESPSW}_{(\xi^*,\eta),G_\gamma}$. Elements of HC_γ and $\mathrm{HESPSW}_{(\xi^*,\eta^*),G_\gamma}$

may be uniquely defined by these elements of "perturbate base". The homomorphism Φ corresponds to φ^E under this type of perturbation operation, so it suffices to note Theorem VI.8.

$\square$

CHAPTER VII

Behavior of Holomorphic 1-Chains near the Perspective Line, as Encoded by Φ

A general theme of the previous sections is that certain classes of holomorphic 1-chains bounded by γ within $\mathbb{CP}^2$ have an isomorphic correspondence to certain classes of local formal "decompositions" of G_γ. At its heart, this isomorphic correspondence transforms the behavior, local to the chosen perspective line $\{w_2 = \xi^* w_0 + \eta^* w_1\}$, of a given holomorphic 1-chain into a formal linear "decomposition" of G_γ locally about (ξ^*, η^*). The first encounter with this theme was the appearance of the isomorphism φ of Theorem IV.6. This was an important forerunning result, but it possessed some quirks which were noted in Chapter VI. For one, the class of holomorphic 1-chains in the domain of φ was dependent on the choice of (ξ^*, η^*). Theorem VI.9 provided the isomorphism Φ, an extension of φ not subject to the fore-mentioned quirks. In particular its domain is HC_γ, independent of (ξ^*, η^*). This sections constitutes a study of the isomorphism Φ and the pursuant applications in an examination of HC_γ.

In a similar fashion to before, let γ be a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \mathbb{CP}^2$. For any fixed $(\xi^*, \eta^*) \in \mathcal{U}_\gamma$, Φ gives a isomorphism from HC_γ to $\mathrm{HESPSW}_{(\xi^*,\eta^*),G_\gamma}$.

First we define notions of degree on the elements of HC_γ and $\mathrm{HESPSW}_{(\xi^*,\eta^*),G_\gamma}$

and show that these notions are in fact preserved by Φ.

Let V be an element of HC_γ. We may decompose V by its positive and negative components, so that $V = V^+ - V^-$, where V^+ and V^- are positive holomorphic 1-chains having support contained in the support of V. Let W be an analytic variety in $\mathbb{CP}^2$ (hence an algebraic variety) avoiding $\operatorname{spt}\gamma$. Define the *degree of positive intersections of* V *with* W to be the total intersection degree of V^+ and W. Likewise define the *degree of negative intersections of* V *with* W to be the total intersection degree of V^- and W.

Let $[P]/[Q]$ be an element of $\mathrm{HESPSW}_{(\xi^*,\eta^*),G}$, where P and Q are members of $\mathrm{RHESPSW}^+_{(\xi^*,\eta^*)}$ which one may recall is a subring of a polynomial ring. The representation of $[P]/[Q]$ is not unique, but choose a representation such that the polynomial degree of P is minimal (the corresponding Q will automatically have minimal polynomial degree). With P and Q so chosen, define the *degree of positive terms* of $[P]/[Q]$ to be the polynomial degree of P and the *degree of negative terms* of $[P]/[Q]$ to be the polynomial degree of Q.

With (ξ^*, η^*) fixed, we will use intersection with the perspective line, $\{w_2 = \xi^* w_0 + \eta^* w_1\}$, to determine degree on HC_γ. By noting the proof of Lemma VI.7, which provides the definition of Φ, it follows that Φ preserves degree.

Theorem VII.1. *Let γ be a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \mathbb{CP}^2$. Let $V \in HC_\gamma$ and $(\xi^*, \eta^*) \in \mathfrak{U}_\gamma$. The degree of positive (resp. negative) terms of $\Phi(V)$ equals the degree of positive (resp. negative) intersections of V with $\{w_2 = \xi^* w_0 + \eta^* w_1\}$.*

We also can derive results by noting how Φ encodes the structure of an analytic variety near the perspective line. Theorem V.5 was one example of this. In that case the required behavior near the selected perspective line included no intersections at the line at ∞ and we were able to finesse the required behavior to include local transversality. As a result, it did not require the full generality of Theorem VI.9 and was able to only use Theorem IV.6. Whenever the behavior we wish to impose does not include the associated restrictions of Theorem IV.6 we might be able to use Theorem IV.6 by employing some generic perturbations (as Theorem V.5 demonstrates). However a simpler and more elegant approach would be to use the greater generality of Theorem VI.9.

So we revisit the notion of bounding permitting only intersection with the perspective line at a given point with non-tangential contact. This was introduced in Theorem V.5, but this time we examine it in light of Theorem VI.9. A preliminary step towards this is the following.

Theorem VII.2. *Let $V \in HC_\gamma$ and $(\xi^*, \eta^*) \in \mathfrak{U}_\gamma$. V avoids the point $[0:1:\eta^*]$ if and only $\Phi(V)$ may be represented by $P_0^+, P_1^+, \ldots, P_{N+}^+$ and $P_0^-, P_1^-, \ldots, P_{N-}^-$ holomorphic in (ξ, η) on some neighborhood Ω of (ξ^*, η^*), and satisfying the s.h.e.s.p.s.w. equations such that $P_0^\pm\big|_{\eta=\eta^*} \neq 0$.*

Proof: Recall that the P_k^+ (resp. P_k^-) give a homogenized form of the elementary symmetric polynomials of the z_1 coordinates of positive (resp. negative) intersections of V with $\{w_2 = \xi w_0 + \eta w_1\}$. So having $P_0^\pm$ not vanish for $\eta = \eta^*$ implies these elementary symmetric polynomials will be bounded. Thus for ξ near ξ^*, V does not intersect $\{w_2 = \xi w_0 + \eta^* w_1\}$ at the line of ∞, thereby implying V isn't incident upon $[0:1:\eta^*]$.

In reverse, from the construction of P_0 from Lemma VI.7, in particular equation (6.19), we see V avoiding the the point $[0:1:\eta]$ will mean that the constructed representation will satisfy $P_0^\pm|_{\eta=\eta^*} \neq 0$.

□

Now we revisit the issue bounding a holomorphic 1-chain having intersections with the perspective line occurring only with non-tangential contact at a prescribed point.

Theorem VII.3. *Let $V \in HC_\gamma$ and $(\xi^*, \eta^*) \in \mathfrak{U}_\gamma$. V may only intersect $\{w_2 = \xi^* w_0 + \eta^* w_1\}$ at $(0, \xi^*)$ with non-tangential contact if and only $\Phi(V)$ may be represented by $P_0^+, P_1^+, \ldots, P_{N^+}^+$ and $P_0^-, P_1^-, \ldots, P_{N^-}^-$ holomorphic in (ξ, η) on some neighborhood Ω of (ξ^*, η^*), and satisfying the s.h.e.s.p.s.w. equations, such that $P_0^\pm\big|_{\eta=\eta^*} \neq 0$ and $\frac{\partial^\ell}{\partial \xi^\ell} P_k^\pm\Big|_{\xi=\xi^*} = 0$, for $\ell < k$.*

Proof: With Theorem VII.2 we may assume that V is not incident upon the point $[0 : 1 : \eta^*]$, and that the s.h.e.s.p.s.w. representations of $\Phi(V)$ may be given with $P_0^\pm|_{\eta=\eta^*} \neq 0$. By shrinking Ω, if necessary, we may assume that P_0^+ and P_0^- are non-vanishing. Dividing through by $P_0^\pm$, we may assume that $P_0^\pm \equiv 1$.

Also it suffices to establish this theorem in the case of when V is positive. For we decompose V by its positive and negative components to $V = V^+ - V^-$, where V^+ and V^- are positive and don't have common components. Applying the theorem to V^+ and V^- separately yields the theorem in general.

Now let $f_j(\xi, \eta)$ describe the z_1 coordinates of the intersections of V with the line $\{w_2 = \xi w_0 + \eta w_1\}$. For (ξ, η) near (ξ^*, η^*), we must technically think of these as being multi-valued holomorphic functions. Assigning appropriate multiplicities, we can think of describing V near $\{w_2 = \xi^* w_0 + \eta^* w_1\}$ as $\sum_j \mu_j \mathbb{V}(z_1 - f_j(z_2 - \eta^* z_1, \eta^*))$. Define $x = z_2 - \eta^* z_1 - \xi^*$ and $y = z_1$. Then V can be perceived near $\{x = 0\}$ as $\sum_j \mu_j \mathbb{V}(y - f_j(\xi^* + x, \eta^*))$. (Note this relates the general arrangement examined in Appendix A.)

Now define $e_m(x) = P_m(\xi^* + z, \eta^*)$ (recalling $P_0 \equiv 1$), and define $c_m(x) =$

$\sum_j \mu_j f_j(\xi^*+x,\eta^*)^m$. These are, respectively, the elementary symmetric polynomials and sums of powers of the f_j. Now we define their standard generating functions.

$$E_x(t) = \sum_{m=0}^{\infty} e_m(x)t^m \tag{7.1}$$

$$C_x(t) = \sum_{m=0}^{\infty} c_{m+1}(x)t^m \tag{7.2}$$

Recognize the combinatorial identity, $C_x(t) = (E_x)'(-t)/E_x(-t)$, and its counterpart $E_x(t) = \exp(\int_0^z C_x(-\tau)\, d\tau)$.

The theorem will follow by the subsequent equivalences.

By Theorem A.4, V intersects $\{x = 0\}$ (which is $\{w_2 = \xi^* w_0 + \eta^* w_1\}$) at $(x, y) = (0, 0)$ (which is $(z_1, z_2) = (0, \xi^*)$) with non-tangential contact if and only if $c_m(x)$ is divisible by x^m for all m. This is equivalent to $C_x(t)$ being representable as x times a Taylor series in (xt) using coefficients holomorphic in x locally about $x = 0$, which is then equivalent to $E_x(t)$ being representable as a Taylor series in (xt) using coefficients holomorphic in x locally about $x = 0$. This translates to be the same as $e_m(x)$ being divisible by x^m and to $\frac{\partial^\ell}{\partial \xi^\ell} P_m\Big|_{\xi=\xi^*} = 0$, for $\ell < m$.

$\square$

Consequentially we can give a result concerning boundaries of holomorphic 1-chains within $\mathbb{C} \times \hat{\mathbb{C}}$. Let $\phi : \hat{\mathbb{C}} \times \hat{\mathbb{C}} - - \to \mathbb{CP}^2$ be the birational map defined as $(z_0 : z_1) \times (w_0 : w_1) \mapsto (z_0 w_1 : z_1 w_0 : z_0 w_0)$, or as $(z, w) \mapsto (\frac{z}{w}, \frac{1}{w})$ on the natural affine portions.

Theorem VII.4. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C} \times \mathbb{C}^* \subset \mathbb{C} \times \hat{\mathbb{C}}$, the following are equivalent:*

1. *γ bounds a holomorphic 1-chain within $\mathbb{C} \times \hat{\mathbb{C}}$*

2. $\phi_*\gamma$ *bounds a holomorphic 1-chain within* $\mathbb{CP}^2$ *that may only intersect* $\{w_2 = 0\}$ *at* $(0,0)$ *with non-tangential contact*

3. $\exists$ *some neighborhood* Ω *of* $(0,0)$ *such that* $\exists$ *non-negative integers* N^+ *and* N^- *and functions* $P_k^+(\xi,\eta)$ *for* $0 \le k \le N^+$ *and* $P_k^-(\xi,\eta)$ *for* $0 \le k \le N^-$*, with* $P_0^\pm|_{\eta=\eta^*} \neq 0$ *and* $\frac{\partial^\ell}{\partial\xi^\ell}P_k^\pm\Big|_{\xi=\xi^*} = 0$ *for* $0 \le \ell < k$*, that are defined on* Ω*, analytic in* (ξ,η)*, and satisfy the s.h.e.s.p.s.w. system of differential equations,* $(P_{k+1})_\xi + (P_k)_\eta = \mu P_k, \forall k \ge 0$*,* $(P_0)_\xi = 0$*, for some analytic function* μ*, (*$\mu = \mu^\pm$ *and* $P_k = P_k^\pm$*) (treating* $P^\pm_{N^\pm+1} = 0$*), such that on* Ω*,*

$$\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{z}{w}\frac{d(1-\xi w-\eta z)}{1-\xi w-\eta z}\right) = \frac{\partial^2}{\partial\xi^2}\left(\frac{P_1^+(\xi,\eta)}{P_0^+(\xi,\eta)} - \frac{P_1^-(\xi,\eta)}{P_0^-(\xi,\eta)}\right) \tag{7.3}$$

4. Any connected neighborhood Ω *of* $(0,0)$ *satisfies condition 3*

5. $\exists$ *some neighborhood* Ω *of* $(0,0)$ *such that* $\exists$ *non-negative integers* N^+ *and* N^- *and functions* $e_k^+(\xi,\eta)$ *for* $1 \le k \le N^+$ *and* $e_k^-(\xi,\eta)$ *for* $1 \le k \le N^-$*, with* $\frac{\partial^\ell}{\partial\xi^\ell}e_k^\pm\Big|_{\xi=\xi^*} = 0$ *for* $0 \le \ell < k$*, that are defined on* Ω*, analytic in* (ξ,η)*, and satisfy the e.s.p.s.w. system of differential equations,* $(e_{k+1})_\xi + (e_k)_\eta = (e_1)_\xi e_k, \forall k \ge 1$*, (*$e_k = e_k^\pm$*) (treating* $e^\pm_{N^\pm+1} = 0$*), such that on* Ω*,*

$$\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{z}{w}\frac{d(1-\xi w-\eta z)}{1-\xi w-\eta z}\right) = \frac{\partial^2}{\partial\xi^2}\left(e_1^+(\xi,\eta) - e_1^-(\xi,\eta)\right) \tag{7.4}$$

Proof:

Condition 1 is equivalent to 2 by the discussion prior to Theorem V.5, notably with $\phi = \phi_0$. Conditions 2 and 3 are equivalent by Theorem VII.3 and a basic integral calculation using the push-forward. By similar reasoning and using Lemma VI.7 we see that $2 \implies 4$. The implications $4 \implies 3$ and $5 \implies 3$ are trivial.

To conclude we show that 3 $\implies$ 5. Assume 3 and shrink the given Ω so that $P_0^{\pm}$ does not vanish on Ω. Setting $e_k^{\pm} = \frac{P_k^{\pm}}{P_0^{\pm}}$ will then yield 3.

$\square$

To conclude this section we give an extension of the normal characterization question, which we term the *characterization of boundaries of holomorphic chains with qualifications*. So choose $(\xi^*, \eta^*) \in \mathfrak{U}_\gamma$ and some $Q_{(\xi^*,\eta^*)}$, a qualification or local property for holomorphic 1-chains near the perspective line $\{w_2 = \xi^* w_0 + \eta^* w_1\}$. One question of interest is knowing when does γ bound a holomorphic 1-chain within $\mathbb{CP}^2$ subject to the qualification $Q_{(\xi^*,\eta^*)}$. If we can determine a property $P_{(\xi^*,\eta^*)}$ on elements of $\mathrm{HESPSW}_{(\xi^*,\eta^*),G_\gamma}$ such that $V \in \mathrm{HC}_\gamma$ has qualification $Q_{(\xi^*,\eta^*)}$ if and only if $\Phi(V)$ has property $P_{(\xi^*,\eta^*)}$, then we may answer this question in a fashion similar to Theorem VI.6.

Several of the results of this section center around correspondences between particular qualifications on holomorphic 1-chains and properties on their image under Φ. We introduce the following theorem to serve as a general template in which any of these results could be placed.

Theorem VII.5. *Let γ be a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \mathbb{CP}^2$. Let $(\xi^*, \eta^*) \in \mathfrak{U}_\gamma$, and assume that $V \in \mathrm{HC}_\gamma$ has qualification $Q_{(\xi^*,\eta^*)}$, depending only on behavior local to $\{w_2 = \xi^* w_0 + \eta^* w_1\}$, if and only if $\Phi(V)$ has property $P_{(\xi^*,\eta^*)}$. The following are equivalent:*

1. *γ bounds a holomorphic 1-chain within $\mathbb{CP}^2$ subject to qualification $Q_{(\xi^*,\eta^*)}$.*

2. *$\exists$ some neighborhood Ω of (ξ^*, η^*) in $\mathfrak{U}_\gamma$ such that $\exists$ non-negative integers N^+ and N^- and functions $P_k^+(\xi, \eta)$ for $0 \le k \le N^+$ and $P_k^-(\xi, \eta)$ for $0 \le k \le N^-$, with $P_0^{\pm} \not\equiv 0$, that are defined on Ω, analytic in (ξ, η), and satisfy the*

s.h.e.s.p.s.w. system of differential equations, $(P_{k+1})_\xi + (P_k)_\eta = \mu P_k, \forall k \geq 0$, $(P_0)_\xi = 0$*, for some analytic function* μ*,* ($\mu = \mu^\pm$ *and* $P = P_k^\pm$) *(treating* $P^\pm_{N^\pm+1} = 0$*), such that on* Ω*,*

$$\frac{\partial^2}{\partial \xi^2} G_\gamma(\xi,\eta) = \frac{\partial^2}{\partial \xi^2}\left(\frac{P_1^+(\xi,\eta)}{P_0^+(\xi,\eta)} - \frac{P_1^-(\xi,\eta)}{P_0^-(\xi,\eta)}\right), \tag{7.5}$$

and furthermore the s.h.e.s.p.s.w. decomposition of G_γ *given by* $P_0^+, P_1^+, \ldots, P_N^+$ *and* $P_0^-, P_1^-, \ldots P_N^-$ *satisfies property* $P_{(\xi^*,\eta^*)}$*.*

3. $\exists$ *some polydisc neighborhood (with respect to the coordinates* (ξ,η)*)* $\Omega \subseteq \mathfrak{U}_\gamma$ *of* (ξ^*,η^*) *which satisfies condition 2 with the additional restriction that* $\mu(\xi,\eta)|_{\xi=\xi^*}$ *is identically zero as a function in* η*.*

CHAPTER VIII

A Focus on the Case of $N^- = 0$

Let μ be an analytic function in (ξ, η), defined in some polydisc (with respect to the coordinate directions ξ and η) Ω that is a neighborhood of (ξ^*, η^*). Define $\mathcal{O}(\Omega)$ to be the ring of analytic functions on Ω. For $N \geq 0$, define the statement "μ satisfies condition $(*_N)$" to mean there exist $P_0, P_1, \ldots, P_N \in \mathcal{O}(\Omega)$, with $P_0 \neq 0$ and defining $P_{N+1} = 0$ that satisfy

$$(8.1) \qquad (P_{k+1})_\xi = \mu P_k - (P_k)_\eta, \text{ for } 0 \leq k \leq N, \qquad (P_0)_\xi = 0.$$

By the discussion preceding Theorem VI.6, we may arbitrarily modify $\mu|_{\xi=\xi^*}$ while leaving μ_ξ unchanged and preserving whether or not μ satisfies condition $(*_N)$. The main result of this section is Theorem VIII.7, which demonstrates a condition on μ_ξ equivalent to μ satisfying $(*_N)$. Of further note this condition on μ_ξ is equivalent to a system of integro-differential equations on μ_ξ. (Define the operator D_ξ to be differentiation with respect to ξ. Define the operator $\int_\xi$ to be anti-differentiation with respect to ξ with base point ξ^* (i.e. $\int_\xi f(\xi, \eta) = \int_{\xi^*}^{\xi} f(\xi', \eta)\, d\xi'$). Do so similarly for η. We call an equation an integro-differential equation on f if it can be expressed using only f, scalars, basic arithmetic operations, and the previously given differential and anti-differential operators.)

To motivate and direct this section, let's consider the case when γ satisfies Theorem VII.1 with $N^- = 0$ and $N^+ = N$ at (ξ^*, η^*). Note that

$$(G_\gamma)_{\xi\xi} = \frac{P_1^+}{P_0^+} = \frac{(\mu^+ P_0^+ - (P_0^+)_\eta)_\xi}{P_0^+} = \mu_\xi^+, \tag{8.2}$$

completely determines μ_ξ^+ and μ^+ satisfies condition $(*_N)$. So the issue of whether or not γ bounds with $N^- = 0$ and $N^+ = N$ is equivalent to whether or not any μ such that $\mu_\xi = (G_\gamma)_{\xi\xi}$ satisfies condition $(*_N)$. This explains why our interest will be centered on μ_ξ and why $\mu|_{\xi=\xi^*}$ will be considered extraneous in evaluating whether μ satisfies $(*_N)$.

Now we begin our technical development of this section. First we define $\mathfrak{U}$ to be the algebra of formal differential expressions on μ_ξ. By this we specifically mean the free $\mathbb{Z}$-algebra with formal generators $\{\boldsymbol{D_\eta^i D_\xi^j \mu}\}_{i\geq 0, j\geq 1}$. (This algebra is constructed as the formal $\mathbb{Z}$-linear combinations of formal products of the given symbols.) For a given μ with domain Ω there exists a natural homomorphism of $\mathfrak{U}$ into the $\mathcal{O}(\Omega)$, which is given by mapping each formal symbol $\boldsymbol{D_\eta^i D_\xi^j \mu}$ to the analytic function $D_\eta^i D_\xi^j \mu$ determined from the specifically given μ. We'll denote this ($\mathbb{Z}$-algebra) homomorphism by a_μ, or refer to it as evaluation using μ. There are naturally defined formal maps $\boldsymbol{D_\xi}$ and $\boldsymbol{D_\eta}$ on $\mathfrak{U}$ that agree with their namesakes under evaluation using any μ. (In other words $a_\mu \circ \boldsymbol{D_\xi} = D_\xi \circ a_\mu$.)

Define $\mathfrak{V}_K$, for $K \geq 0$, to be the free $\mathfrak{U}$-module generated by formal elements $\{\boldsymbol{D_\xi^j P_i}\}_{0\leq j\leq i\leq K}$. (The elements of $\mathfrak{V}_K$ are formal $\mathfrak{U}$-linear combinations of the symbols $\{\boldsymbol{D_\xi^j P_i}\}_{0\leq j\leq i\leq K}$.) Define $\mathfrak{V}_{-1}$ to be the zero module. Given functions μ and $P_0, P_1, \ldots, P_K$ in $\mathcal{O}(\Omega)$, there is a natural ($\mathbb{Z}$-module) homomorphism from $\mathfrak{V}_K$ to $\mathcal{O}(\Omega)$, which is given by evaluating each formal symbol according to the given choice of μ and $P_0, P_1, \ldots, P_K$. We'll denote this homomorphism by $b_{\mu,P_0,\ldots,P_K}$ or refer to it as evaluation using μ and $P_0, P_1, \ldots, P_K$. (To help conceptualize this notation one may consider $b_{\mu,P_0,\ldots,P_K}(\mathbf{x})$ as meaning $\mathbf{x}|_{\boldsymbol{\mu}=\mu, \boldsymbol{P_0}=P_0,\ldots,\boldsymbol{P_K}=P_K}$.)

The rationale of introducing $\mathfrak{V}_K$ stems from the following lemma.

Lemma VIII.1. *For $\ell > k \geq 0$, there exists a $p_{k,\ell} \in \mathfrak{V}_{k-1}$, such that for μ and $P_0, P_1, P_2, \ldots, P_k$ in $\mathcal{O}(\Omega)$ that satisfy (8.1) with $N = k-1$, then they must satisfy $D_\xi^\ell P_k = b_{\mu,P_0,\ldots,P_{k-1}}(p_{k,\ell})$.*

Proof: We'll first establish the lemma for the case $\ell = k+1$ by induction on k. For $k = 0$, the statement is trivially true as $(P_0)_\xi = 0$. In case one wishes to see a less trivial base case, one can establish $k = 1$ by noting via an easy manipulation that $(P_1)_{\xi\xi} = \frac{\partial}{\partial\xi}(\mu P_0 - (P_0)_\eta) = \mu_\xi P_0$.

We assume we have established it for all k up to and including k', and now set about to establish it for $k = k'+1$.

First we provide this ancillary fact. For m, n such that $0 \leq m \leq k-1 = k'$, $n \geq 0$,

$$
\begin{aligned}
D_\xi^{n+1} P_{m+1} &= D_\xi^n(\mu P_m - (P_m)_\eta) \\
&= \sum_{j=0}^{n-1} \left[\binom{n}{j} (D_\xi^{n-j}\mu)(D_\xi^j P_m) \right] + (\mu - D_\eta)\left(D_\xi^n P_m\right).
\end{aligned} \tag{8.3}
$$

One consequence of this is that

$$
D_\xi^{k'+2} P_{k'+1} = \sum_{j=0}^{k'} \left[\binom{k'+1}{j} (D_\xi^{k'+1-j}\mu)(D_\xi^j P_{k'}) \right] + (\mu - D_\eta)\left(D_\xi^{k'+1} P_{k'}\right).
$$

Note that the summation term, when viewed a formal expression, is in $\mathfrak{V}_{k'}$, so our only remaining concern is the rightmost term, which, by the inductive hypothesis, may be represented as a $\mathbb{Z}$-linear combination of terms of the form $(\mu - D_\eta)(rD_\xi^n P_m)$, where $r \in \mathfrak{U}$ and $0 \leq n \leq m \leq k'-1$. So it suffices to show a term of this form can be represented as an expression from $\mathfrak{V}_{k'}$. Now note

$$
\begin{aligned}
(\mu - D_\eta)(rD_\xi^n P_m) &= r(\mu - D_\eta)(D_\xi^n P_m) - (D_\eta r) D_\xi^n P_m \\
&= rD_\xi^{n+1} P_{m+1} - \sum_{j=0}^{n-1} \left[\left(\binom{n}{j} r(D_\xi^{n-j}\mu) \right) (D_\xi^j P_m) \right] - (D_\eta r) D_\xi^n P_m,
\end{aligned} \tag{8.4}
$$

where the second equality makes use of the previous ancillary fact. This prescribes a formal expression in $\mathfrak{V}_{k'}$, and concludes the inductive step.

The lemma's statement is now established when $\ell = k+1$. Now we define $\boldsymbol{D_\xi}$, a self map of $\mathfrak{V}_k$ which behaves as D_ξ under evaluation for any μ and $P_0, P_1, \ldots, P_k$ that are subject to (8.1) with $N = k-1$. (To be precise we mean that $b_{\mu,P_0,\ldots,P_k} \circ \boldsymbol{D_\xi} = D_\xi \circ b_{\mu,P_0,\ldots,P_k}$.) As $\mathfrak{U}$ has $\boldsymbol{D_\xi}$ naturally defined, we only need be concerned with defining $\boldsymbol{D_\xi}$ on the module generators of $\mathfrak{V}_k$. For the symbols $\boldsymbol{D_\xi^j P_i}$, where $j < i$ we choose the natural definition for $\boldsymbol{D_\xi}$. As the lemma has been established for $\ell = k+1$, for the remaining symbols we define $\boldsymbol{D_\xi}(\boldsymbol{D_\xi^m P_m}) = p_{m,m+1}$. Thus we have completed the definition of $\boldsymbol{D_\xi}$ on $\mathfrak{V}_k$. In conclusion the case $\ell > k+1$ follows by letting $p_{k,\ell} = \boldsymbol{D_\xi}^{\ell-k-1} p_{k,k+1}$.

$\square$

For the sake of the intuitive picture we simply provided an existential proof here. But it is straightforward, though calculationally intensive, to use this proof to guide in forming a constructive proof. As a result of this proof we can define $\rho_{k,\ell,i,j}$ in $\mathfrak{U}$ such that $p_{k,\ell}$ can be given as

$$p_{k,\ell} = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \rho_{k,\ell,i,j}(\boldsymbol{D_\xi^j P_i}). \tag{8.5}$$

Our definition for $\rho_{k,\ell,i,j}$ can be made well-defined for general choices of integers k, ℓ, i, and j, with $k, \ell \geq 0$ such that

$$(D_\xi^\ell P_k) = \sum_i \sum_j \rho_{k,\ell,i,j}(D_\xi^j P_i). \tag{8.6}$$

(At this point and following, we drop the notational distinction between elements of $\mathfrak{V}_k$ or $\mathfrak{U}$ and the expressions they represent.)

A constructive proof for Lemma VIII.1 and the derivation of the definition for $\rho_{k,\ell,i,j}$ is given in the appendix, along with some relations regarding $\rho_{k,\ell,i,j}$. We will simply state here the definition derived there, though first we will make some comments on notation. We routinely make use of the Kronecker delta function, which is given as $\delta_p = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \end{cases}$, where p may be any integer. Also we permit empty summations, which are precisely of the form $\sum_{i=m}^{m-1}$ and which we hold to equal 0. (However we will not use "reverse summations", that is ones of the form $\sum_{i=m}^{m-n}$, where $n \geq 2$.) Also on occasion we might make use of extended binomial coefficients, which define $\binom{n}{m}$ to be 0 when $m < 0$ or $m > n$, given $n \geq 0$. Several combinatorial identities still hold with the extended definition. Since this is considered nonstandard, we will make an explicit note of where we employ this extension of the binomial coefficients. These notational choices will serve to unify a number of the following equations and cases. Now the definition for $\rho_{k,\ell,i,j}$, for $k, \ell \geq 0$ is as follows.

- For $0 \leq \ell \leq k$,

$$\rho_{k,\ell,i,j} = \delta_{k-i}\delta_{\ell-j}.$$

- For $0 \leq k < \ell$ and any of $j < 0$, $i < j$, or $i \geq k$,

$$\rho_{k,\ell,i,j} = 0.$$

- For $0 \leq j \leq i < k < \ell$,

(8.7)
$$\rho_{k,\ell,i,j} = \sum_{j_1=k}^{\ell-2} \binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu)\rho_{k-1,j_1,i,j} - \sum_{j_1=j+1}^{i} \binom{j_1}{j} (D_\xi^{j_1-j}\mu)\rho_{k-1,\ell-1,i,j_1} - (D_\eta \rho_{k-1,\ell-1,i,j}) + \binom{\ell-1}{j} (D_\xi^{\ell-1-j}\mu)\delta_{k-1-i} + \rho_{k-1,\ell-1,i-1,j-1}.$$

Remark: We provide this case breakdown as the way which $\rho_{k,\ell,i,j}$ may be well-defined. But as one note we point out that (8.7) holds true whenever $0 \leq k < \ell$.

As noted near the end of the proof of Lemma VIII.1, we can provide a self-map of $\mathfrak{V}_K$ which agrees with D_ξ under the homomorphism $b_{\mu,P_0,\ldots,P_K}$ when μ and $P_0, P_1, \ldots P_K$ are subject to (8.1) for $N = K-1$. An important ramification of this is that if $P_0, P_1, \ldots, P_N$ satisfy (8.1) with $P_{N+1} = 0$ then they satisfy a system of linear ordinary differential equations with respect to ξ with μ given as fixed. Specifically for $0 \leq k \leq N$

$$D_\xi^{k+1} P_k = p_{k,k+1} = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \rho_{k,k+1,i,j}(D_\xi^j P_i). \tag{8.8}$$

Also they satisfy

$$0 = p_{N+1,N+2} = \sum_{i=0}^{N} \sum_{j=0}^{i} \rho_{N+1,N+2,i,j}(D_\xi^j P_i) \tag{8.9}$$

Now we wish to express (8.8) instead as a system of linear first-order ordinary differential equations with respect to ξ. By using $v_{k,\ell}$ in place of $D_\xi^\ell P_k$, we have that for $0 \leq \ell < k$, $D_\xi v_{k,\ell} = v_{k,\ell+1}$ and for $0 \leq \ell = k$, $D_\xi v_{k,k} = p_{k,k+1} = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \rho_{k,k+1,i,j} v_{i,j}$.

This system of differential equations can be represented in matrix form. While there are any number of ways to do this, we will provide one particularly strategic way. To aid in the technical description, as well as the visual presentation, we make a momentary digression to introduce the following definitions and notations. Let $I = \{(i,j) \mid 0 \leq j \leq i\}$ with the total ordering given by $(i,j) \preceq (i',j')$ if and only if $i < i'$ or $i = i'$ and $j \geq j'$. This ordering is the lexicographical ordering, with the reverse ordering understood in the second entry. (By using subsets of this this index set, the matrices we will later express will possess some triangularity characteristics.) Let $I_N = \{(i,j) \mid 0 \leq j \leq i \leq N\} \subset I$, inheriting a total ordering structure from I. If M is a matrix with rows indexed by A and columns indexed by B, we will use the notation M_α^β to refer to the entry in the row indexed by $\alpha \in A$ and the column indexed by $\beta \in B$.

Let

$$\vec{v}_N = \begin{bmatrix} v_{0,0} & v_{1,1} & v_{1,0} & v_{2,2} & \cdots & v_{2,0} & \cdots & v_{N,N} & \cdots & v_{N,0} \end{bmatrix}^{\mathrm{T}}, \tag{8.10}$$

where $v_{i,j}$ are assumed to be in $\mathcal{O}(\Omega)$. This is a column vector indexed by I_N, and $(\vec{v}_N)_{(i,j)} = v_{i,j}$. Define A_N as the matrix (rows and columns both indexed by I_N given such that the previous system of differential equations on $v_{i,j}$ may be simply given as $D_\xi \vec{v}_N = A_N \vec{v}_N$. This unambiguously defines A_N, with entries in $\mathfrak{U}$. We see that A_N can be defined entry-wise by the following.

- For $(0,0) \preceq (i,j), (i',j') \preceq (N,0)$ with $i \neq j$ and $(i',j') \neq (i,j+1)$,
  $A_N{}_{(i,j)}^{(i',j')} = 0$.
- For $(0,0) \preceq (i,j) \preceq (N,0)$ with $i \neq j$,
  $A_N{}_{(i,j)}^{(i,j+1)} = 1$.
- For $(0,0) \preceq (i,i) \preceq (i',j') \preceq (N,0)$,
  $A_N{}_{(i,i)}^{(i',j')} = 0$
- For $(0,0) \preceq (i',j') \prec (i,i) \preceq (N,0)$,
  $A_N{}_{(i,i)}^{(i',j')} = \rho_{i,i+1,i',j'}$.

By understanding the general definition of $\rho_{k,\ell,i,j}$ (as given in Appendix B), A_N could be more concisely defined for all entries as $A_N{}_{(i,j)}^{(i',j')} = \rho_{i,j+1,i',j'}$.

Now it should be noted that A_N is strictly lower triangular. This is also quite evident from the following visual presentation of A_N.

$$
A_N = \left[\begin{array}{cccccccccccc}
0 & & & & & & & & & & & \\
p_{1,2} & 0 & & & & & & & & & & \\
0 & 1 & 0 & & & & & & & & & \\
\multicolumn{3}{c}{\hrulefill p_{2,3} \hrulefill} & 0 & & & & & & & & \\
0 & 0 & 0 & 1 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 1 & 0 & & & & & & \\
\vdots & & & & & & \ddots & & & & & \\
\multicolumn{7}{c}{\hrulefill p_{N,N+1} \hrulefill} & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & 1 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & & \\
\vdots & & & & & \vdots & & \vdots & \ddots & \ddots & \ddots & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & 0
\end{array}\right] \tag{8.11}
$$

Now we also seek to express (8.9) in this language. This can be given by the equation $M_N \vec{v}_N = 0$, where M_N is an accordingly defined $1 \times I_N$ matrix. In particular M_N is defined entry-wise as ${M_N}^{(i,j)} = \rho_{N+1,N+2,i,j}$, and can be visually represented as

$$
M_N = \left[\hrulefill p_{N+1,N+2} \hrulefill\right]. \tag{8.12}
$$

Next if $P_0, P_1, \ldots, P_N$ satisfy (8.1) with $P_{N+1} = 0$, then the $v_{i,j}$ corresponding to $D_\xi^j P_i$, $0 \le j \le i \le N$, also satisfy a system of linear first-order ordinary differential equations with respect to η. For $0 \le j \le i \le N$,

$$
\begin{aligned}
D_\eta(D_\xi^j P_i) &= D_\xi^j(\mu P_i - (P_{i+1})_\xi) \\
&= \sum_{j'=0}^{j} \left[\binom{j}{j'} (D_\xi^{j-j'} \mu)(D_\xi^{j'} P_i) \right] - D_\xi^{j+1} P_{i+1}.
\end{aligned} \tag{8.13}
$$

So from the above equation (and noting $P_{N+1} = 0$) we can define a matrix B_N such that the above system of differential equations can be given as $D_\eta \vec{v}_N = B_N \vec{v}_N$. We can define B_N entry-wise as follows.

- For $(0,0) \preceq (i',j') \prec (i,j) \preceq (N,0)$,
  ${B_N}_{(i,j)}^{(i',j')} = 0.$
- For $(0,0) \preceq (i,j) \preceq (i,j') \preceq (i,0) \preceq (N,0)$,
  ${B_N}_{(i,j)}^{(i,j')} = \binom{j}{j'}(D_\xi^{j-j'}\mu).$
- For $(0,0) \preceq (i,j) \preceq (i,0) \prec (i',j') \preceq (N,0)$ with $(i',j') \neq (i+1,j+1)$,
  ${B_N}_{(i,j)}^{(i',j')} = 0.$
- For $(0,0) \preceq (i,j) \prec (i+1,j+1) \preceq (N,0)$,
  ${B_N}_{(i,j)}^{(i+1,j+1)} = -1.$

For a more compact formulation of the entry-wise definitions we can define ${B_N}_{(i,j)}^{(i',j')} = \delta_{i-i'}\binom{j}{j'}D_\xi^{j-j'}\mu - \delta_{i+1-i'}\delta_{j+1-j'}$, where we understand the extended definition of binomial coefficients given following Lemma VIII.1.

Note B_N is upper triangular and has entries in $\mathfrak{U}$, with the exception of the diagonal, which has entries μ. B_N is visually demonstrated below.

(8.14)

$$B_N = \left[\begin{array}{c|cc|ccc|c|ccccc}
\mu & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
 & \mu & \mu_\xi & -1 & 0 & 0 & & & & & & \\
 & & \mu & 0 & -1 & 0 & & & & & & \\
\hline
 & & & \mu & 2\mu_\xi & \mu_{\xi\xi} & & 0 & 0 & 0 & \cdots & 0 \\
 & & & & \mu & \mu_\xi & & 0 & 0 & 0 & \cdots & 0 \\
 & & & & & \mu & & 0 & 0 & 0 & \cdots & 0 \\
\hline
 & & & & & & \ddots & & & & & \\
\hline
 & & & & & & & \mu & N\mu_\xi & \binom{N}{2}\mu_{\xi\xi} & \cdots & D_\xi^N \mu \\
 & & & & & & & & \mu & \binom{N-1}{1}\mu_\xi & \cdots & D_\xi^{N-1}\mu \\
 & & & & & & & & & \mu & \cdots & D_\xi^{N-2}\mu \\
 & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & \mu
\end{array}\right]$$

The previous discussion proves the following theorem.

Theorem VIII.2. *If $P_0, P_1, \ldots, P_N$ satisfy (8.1) with $P_{N+1} = 0$ and $v_{i,j} = D_\xi^j P_i$ then $\vec{v}_N \in \ker(D_\xi - A_N) \cap \ker(D_\eta - B_N) \cap \ker(M_N)$.*

A stronger converse is also true.

Theorem VIII.3. *Assume $\vec{v}_N \in \ker(D_\xi - A_N) \cap \ker(D_\eta - B_N)$ and let $P_i = v_{i,0}$ for $0 \leq i \leq N$. $P_0, P_1, \ldots, P_N$ satisfy (8.1) with $P_{N+1} = 0$.*

Proof: From $\vec{v}_N \in \ker(D_\xi - A_N)$ we see that $D_\xi v_{0,0} = 0$ and that for $0 < i \leq N$, $D_\xi v_{i,0} - v_{i,1} = 0$. From $\vec{v}_N \in \ker(D_\eta - B_N)$ we see that for $0 \leq i < N$, $D_\eta v_{i,0} - \mu v_{i,0} + v_{i+1,1} = 0$ and that $D_\eta v_{N,0} - \mu v_{N,0} = 0$. These together show that the P_i as defined will satisfy (8.1) with $P_{N+1} = 0$.

$\square$

The importance of $\ker(M_N)$ is that $D_\xi - A_N$ and $D_\eta - B_N$ will commute exactly on $\ker M_N$. To show this we compute the commutator of $D_\xi - A_N$ and $D_\eta - B_N$.

Lemma VIII.4. *Let Δ_N be the I_N indexed column vector that is entry-wise defined as $\Delta_{N(i,j)} = \delta_{i-N}\delta_{j-N}$. Then*

$$[D_\xi - A_N, D_\eta - B_N] = -\Delta_N M_N. \tag{8.15}$$

Proof: By basic operations note

$$\begin{aligned}[D_\xi - A_N, D_\eta - B_N] &= [D_\xi, D_\eta] - [A_N, D_\eta] - [D_\xi, B_N] + [A_N, B_N] \\ &= (A_N)_\eta - (B_N)_\xi + A_N B_N - B_N A_N.\end{aligned} \tag{8.16}$$

So now we focus on the entry-wise calculation of each of these terms, using the more condensed entry-wise definitions to prevent the proliferation of cases. Namely we will use that ${B_N}_{(i,j)}^{(i',j')} = \delta_{i-i'}\binom{j}{j'}D_\xi^{j-j'}\mu - \delta_{i+1-i'}\delta_{j+1-j'}$ for general (i,j) and (i',j'), ${A_N}_{(i,j)}^{(i',j')} = \delta_{i-i'}\delta_{j+1-j'}$, for $0 \le j < i$ and general (i',j'), and ${A_N}_{(i,i)}^{(i',j')} = \rho_{i,i+1,i',j'}$, for general i and (i',j'). (We are using Kronecker delta and generalized binomial coefficient notation.)

For general (i,j) and (i',j') note that

$$(D_\xi B_N)_{(i,j)}^{(i',j')} = \delta_{i-i'}\binom{j}{j'}(D_\xi^{j-j'+1}\mu) \tag{8.17}$$

For the remaining terms, calculation is facilitated by breaking into cases. Now we consider the case for (i,j) such that $0 \le j < i \le N$ and that (i',j') is general (i.e. $0 \le j' \le i' \le N$). Then we point out the following calculations with this case assumed.

$$(D_\eta A_N)^{(i',j')}_{(i,j)} = 0 \tag{8.18}$$

$$\begin{aligned}(A_N B_N)^{(i',j')}_{(i,j)} &= \sum_{i_1=0}^{N} \sum_{j_1=0}^{i_1} \delta_{i-i_1}\delta_{j+1-j_1} {B_N}^{(i',j')}_{(i_1,j_1)} = {B_N}^{(i',j')}_{(i,j+1)} \\ &= \delta_{i-i'} \binom{j+1}{j'} (D_\xi^{j+1-j'}\mu) - \delta_{i+1-i'}\delta_{j+2-j'}\end{aligned} \tag{8.19}$$

$$\begin{aligned}(B_N A_N)^{(i',j')}_{(i,j)} &= \sum_{i_1=0}^{N} \sum_{j_1=0}^{i_1} \left(\delta_{i-i_1} \binom{j}{j_1} (D_\xi^{j-j_1}\mu) - \delta_{i+1-i_1}\delta_{j+1-j_1} \right) {A_N}^{(i',j')}_{(i_1,j_1)} \\ &= \sum_{j_1=0}^{j} \binom{j}{j_1} (D_\xi^{j-j_1}\mu) {A_N}^{(i',j')}_{(i,j_1)} - \begin{cases} {A_N}^{(i',j')}_{(i+1,j+1)} & \text{if } i < N \\ 0 & \text{if } i = N \end{cases} \\ &= \delta_{i-i'} \binom{j}{j'-1} (D_\xi^{j-j'+1}\mu) - \delta_{i+1-i'}\delta_{j+2-j'}\end{aligned} \tag{8.20}$$

So the calculations of these terms in this particular case yields from that for $0 \le j < i \le N$ and $0 \le j' \le i' \le N$,

$$[D_\xi - A_N, D_\eta - B_N]^{(i',j')}_{(i,j)} = 0 \tag{8.21}$$

Now we consider the remaining case, that is assuming that $0 \le j = i \le N$ and $0 \le j' \le i' \le N$. We calculate the following assuming this case.

$$(D_\eta A_N)^{(i',j')}_{(i,i)} = D_\eta \rho_{i,i+1,i',j'} \tag{8.22}$$

$$(8.23)\quad (A_N B_N)_{(i,i)}^{(i',j')} = \sum_{i_1=0}^{N}\sum_{j_1=0}^{i_1} A_N{}_{(i,i)}^{(i_1,j_1)}\left(\delta_{i_1-i'}\binom{j_1}{j'}(D_\xi^{j_1-j'}\mu) - \delta_{i_1+1-i'}\delta_{j_1+1-j'}\right)$$

$$= \sum_{j_1=0}^{i'} \rho_{i,i+1,i',j_1}\binom{j_1}{j'}(D_\xi^{j_1-j'}\mu) - \begin{cases} \rho_{i,i+1,i'-1,j'-1} & \text{if } j'>0 \\ 0 & \text{if } j'=0 \end{cases}$$

$$= \sum_{j_1=j'+1}^{i'} \rho_{i,i+1,i',j_1}\binom{j_1}{j'}(D_\xi^{j_1-j'}\mu) + \rho_{i,i+1,i',j'}\mu - \rho_{i,i+1,i'-1,j'-1}$$

$$(8.24)\quad (B_N A_N)_{(i,j)}^{(i',j')}$$

$$= \sum_{i_1=0}^{N}\sum_{j_1=0}^{i_1-1}\left(\delta_{i-i_1}\binom{i}{j_1}(D_\xi^{i-j_1}\mu) - \delta_{i+1-i_1}\delta_{i+1-j_1}\right)\delta_{i_1-i'}\delta_{j_1+1-j'}$$

$$+ \sum_{i_1=0}^{N}\left(\delta_{i-i_1}\binom{i}{i_1}(D_\xi^{i-i_1}\mu) - \delta_{i+1-i_1}\delta_{i+1-i_1}\right)\rho_{i_1,i_1+1,i',j'}$$

$$= \delta_{i-i'}\binom{i}{j'-1}(D_\xi^{i-j'+1}\mu) - \begin{cases} \delta_{i+1-i'}\delta_{i-j'+2} & \text{if } j'>0 \\ 0 & \text{if } j'=0 \end{cases}$$

$$+ \mu\rho_{i,i+1,i',j'} - \begin{cases} \rho_{i+1,i+2,i',j'} & \text{if } i<N \\ 0 & \text{if } i=N \end{cases}$$

$$= \delta_{i-i'}\binom{i}{j'-1}(D_\xi^{i-j'+1}\mu) + \mu\rho_{i,i+1,i',j'} - \rho_{i+1,i+2,i',j'} + \delta_{i-N}\rho_{i+1,i+2,i',j'}$$

The calculation of these terms then yields that for $0 \le j = i \le N$ and $0 \le j' \le i' \le N$,

$$(8.25)\quad [D_\xi - A_N, D_\eta - B_N]_{(i,i)}^{(i',j')} = -\delta_{i-N}\rho_{N+1,N+2,i',j'}$$

$$+ \rho_{i+1,i+2,i',j'} - \rho_{i,i+1,i'-1,j'-1} - \delta_{i-i'}\binom{i+1}{j'}(D_\xi^{i-j'+1}\mu)$$

$$+ \sum_{j_1=j'+1}^{i'}\binom{j_1}{j'}(D_\xi^{j_1-j'}\mu)\rho_{i,i+1,i',j_1} + D_\eta\rho_{i,i+1,i',j'}$$

$$= -\delta_{i-N}M_N{}^{(i',j')},$$

where the last equality holds by the relation on $\rho_{k,\ell,i,j}$ given in (8.7).

As a result of both cases of calculations, the lemma holds.

$\square$

Now we turn our attention onto the systems of linear differential equations represented by the operators $D_\xi - A_N$ and $D_\eta - B_N$. Define $\tilde{\Omega} = \{\eta \in \mathbb{C} \mid (\xi^*, \eta) \in \Omega\}$, which is a disc in $\mathbb{C}$ containing η^*. Define $\tilde{\mathfrak{U}}$ to be the free algebra generated formally by $\{(D_\eta^i D_\xi^j \mu)|_{\xi=\xi^*}\}_{i\geq 0, j\geq 1}$, which has a natural homomorphism for a given μ into the algebra of analytic functions on $\tilde{\Omega}$. Now let

$$\vec{w}_N = \begin{bmatrix} w_{0,0} & w_{1,1} & w_{1,0} & w_{2,2} & \cdots & w_{2,0} & \cdots & w_{N,N} & \cdots & w_{N,0} \end{bmatrix}^{\mathrm{T}}, \tag{8.26}$$

where $w_{i,j}$ are in $\mathcal{O}(\tilde{\Omega})$.

The equation $D_\xi \vec{v}_N = A_N \vec{v}_N$ gives a system of linear first order ordinary differential equations (with respect to ξ) that is analytically parameterized by η. Then by the fundamental properties of linear first order ordinary differential equations, for any given set of initial conditions at $\xi = \xi^*$ there exists an unique solution to $D_\xi \vec{v}_N = A_N \vec{v}_N$, and furthermore the solution $\vec{v}$ linearly depends on the initial conditions at $\xi = \xi^*$. This means there exists a unique matrix K_N (dependent on μ) such that $\vec{v}_N = K_N \vec{w}_N$ is the solution to $D_\xi \vec{v}_N = A_N \vec{v}_N$ with initial conditions $\vec{w}_N$ at $\xi = \xi^*$. Note the entries of K_N are analytic functions with domain Ω. Also by the strict lower triangularity of A_N, K_N is lower triangular with 1's on the diagonal. Note $(K_N)_\xi \vec{w}_N = D_\xi(K_N \vec{w}_N) = A_N(K_N \vec{w}_N)$ for all $\vec{w}_N$, so

$$(K_N)_\xi = A_N K_N \tag{8.27}$$

And by restricting the equation $\vec{v}_N = K_N \vec{w}_N$ to $\xi = \xi^*$, we see that

$$K_N|_{\xi=\xi^*} = \mathrm{Id}. \tag{8.28}$$

So in fact we could choose to define K_N by equations (8.27) and (8.28).

Note that K_N is invertible and prescribes a particular change of coordinates on $(\mathcal{O}(\Omega))^{I_N}$. This leads to the following theorem.

Theorem VIII.5.

$$K_N^{-1}(\ker(D_\xi - A_N) \cap \ker(D_\eta - B_N) \cap \ker(M_N)) = \ker(D_\xi) \cap \ker((D_\eta - B_N)K_N) \cap \ker(M_N K_N) \tag{8.29}$$

Proof: The proof is straightforward from the following, which uses the invertibility of K_N and (8.27).

$$\ker((D_\xi - A_N)K_N) = \ker(K_N D_\xi) = \ker(D_\xi) \tag{8.30}$$

$\square$

So after a change of coordinates given by K_N, it is suitable to restrict to $\xi = \xi^*$. Define $\tilde{B}_N = B_N|_{\xi=\xi^*}$, which by choosing $\tilde{\mu} \equiv 0$, is a strictly upper triangular matrix with entries in $\tilde{\mathfrak{U}}$. Note that $((D_\eta - B_N)K_N)|_{\xi=\xi^*} = (D_\eta - \tilde{B}_N)$ is a self-operator on $(\mathcal{O}(\tilde{\Omega}))^{I_N}$

Now in a fashion similar to before, define

$$\vec{y}_N = \begin{bmatrix} y_{0,0} & y_{1,1} & y_{1,0} & y_{2,2} & \cdots & y_{2,0} & \cdots & y_{N,N} & \cdots & y_{N,0} \end{bmatrix}^{\mathrm{T}}, \tag{8.31}$$

where $y_{i,j}$ are complex numbers. Similarly, there exists a unique matrix L_N (dependent on μ) such that $\vec{w}_N = L_N \vec{y}_N$ gives the solution to $D_\eta \vec{w}_N = \tilde{B}_N \vec{w}_N$ with initial conditions $\vec{y}$ at $\eta = \eta^*$. The entries of L_N are analytic functions with domain $\tilde{\Omega}$. By the strict upper triangularity of $\tilde{B}_N$, L_N is also upper triangular with 1's on the diagonal. Note $(L_N)_\eta \vec{y}_N = D_\eta(L_N \vec{y}_N) = \tilde{B}_N L_N \vec{y}_N$ for all $\vec{y}_N$, so

$$(L_N)_\eta = \tilde{B}_N L_N \tag{8.32}$$

And by evaluating $\vec{w}_N = L_N \vec{y}_N$ at $\eta = \eta^*$, we derive that

$$L_N|_{\eta=\eta^*} = \mathrm{Id}. \tag{8.33}$$

So L_N could have been independently defined according to equations (8.32) and (8.33).

Note that L_N is invertible and suggests a change of variables, which produces the following key theorem.

Theorem VIII.6.

$$\begin{aligned} L_N^{-1} K_N^{-1}(\ker(D_\xi - A_N) \cap \ker(D_\eta - B_N) \cap \ker(M_N)) \\ = \ker(D_\xi) \cap \ker(D_\eta) \cap \ker(M_N K_N L_N) \end{aligned} \tag{8.34}$$

Proof: As L_N is invertible and constant with respect to ξ, it holds that

$$\ker(D_\xi L_N) = \ker(L_N D_\xi) = \ker(D_\xi).$$

So using Theorem VIII.5 yields that

$$\begin{aligned} L_N^{-1} K_N^{-1}(\ker(D_\xi - A_N) \cap \ker(D_\eta - B_N) \cap \ker(M_N)) \\ = \ker(D_\xi) \cap \ker((D_\eta - B_N) K_N L_N) \cap \ker(M_N K_N L_N) \end{aligned} \tag{8.35}$$

Assume $\vec{z}$ is an element of the set given in (8.35). Then note

$$0 = ((D_\eta - B_N) K_N L_N \vec{z})|_{\xi=\xi^*} = (D_\eta - \tilde{B}_N) L_N \vec{z} = L_N D_\eta \vec{z}. \tag{8.36}$$

Thus $\vec{z}$ is a member of the RHS of (8.34).

Now assume $\vec{z}$ is a member of the RHS of (8.34). Observe, using Lemma VIII.4 and the definition of K_N, that

$$(D_\xi - A_N)(D_\eta - B_N)K_N L_N \vec{z} = (D_\eta - B_N)(D_\xi - A_N)K_N L_N \vec{z} = 0 \tag{8.37}$$

Using the definitions of K_N and L_N we get that

$$\begin{aligned}(D_\eta - B_N)K_N L_N \vec{z} &= K_N \left(((D_\eta - B_N)K_N L_N \vec{z})|_{\xi=\xi^*} \right) \\ &= K_N(D_\eta - \tilde{B}_N)L_N \vec{z} = 0\end{aligned} \tag{8.38}$$

□

So now we answer the original question of this section.

Theorem VIII.7. *Let $\mu(\xi, \eta)$ be an analytic function on some polydisc neighborhood Ω of (ξ^*, η^*) that vanishes when $\xi = \xi^*$ and let N be a fixed non-negative integer. The square matrices K_N and L_N (derived from the square matrices A_N and B_N) and row vector M_N are all defined (dependently on μ) by the previous discussion, most directly (8.11), (8.12), (8.14), (8.27), (8.28), (8.32), and (8.33). μ satisfies condition $(*_N)$ (which means there exists $P_0, P_1, \ldots, P_N$ analytic functions on Ω that satisfy (8.1) with $P_{N+1} = 0$) if and only if the entries of the row vector $M_N K_N L_N$ are linearly dependent over $\mathbb{C}$.*

Proof: By Theorem VIII.2 and Theorem VIII.3 we see that the existence of such $P_0, P_1, \ldots, P_N$ is equivalent to the complex linear space $\ker(D_\xi - A_N) \cap \ker(D_\eta - B_N) \cap \ker(M_N)$ being non-trivial (positive dimensional). By the change of variables of Theorem VIII.6 the non-triviality of this space is equivalent to the non-triviality of $\ker(D_\xi) \cap \ker(D_\eta) \cap \ker(M_N K_N L_N)$. This is equivalent to the existence of a constant complex-valued column vector in the kernel of $M_N K_N L_N$ which is equivalent to the linear dependence of the entries of $M_N K_N L_N$ over $\mathbb{C}$.

$\square$

Remark: K_N and L_N can be integro-differentially expressed in terms of μ, and M_N can be expressed differentially in terms of μ. Also the appendix provides a (sharp) means of determining the linear dependence of analytic functions in several variables. This then shows that condition $(*_N)$ on μ is equivalent to an integro-differential condition on μ.

Theorem VIII.6 is also pertinent to bounding in $\mathbb{CP}^2$ while avoiding a line with exception of non-tangential contact at one prescribed point, which was discussed in Chapter V. In particular note Theorem VII.3, which motivates the following theorem.

Theorem VIII.8. *Let $\mu(\xi,\eta)$ be an analytic function on some polydisc neighborhood Ω that vanishes when $\xi=\xi^*$ and let N be a fixed non-negative integer. Let square matrices K_N and L_N and the row vector M_N be defined dependently on μ as in Theorem VIII.7. There exist analytic functions $P_0, P_1, \ldots, P_N$ on Ω that satisfy (8.1) with $P_{N+1}=0$ and $D_\xi^\ell P_k|_{\xi=\xi^*}=0$ for $0\le \ell < k \le N$ if and only if $(M_N K_N L_N)^{(0,0)}$, $(M_N K_N L_N)^{(1,1)}$, …, $(M_N K_N L_N)^{(N,N)}$ are linearly dependent over $\mathbb{C}$.*

Moreover, there exist such analytic functions $P_0, P_1, \ldots, P_N$ on Ω that additionally satisfy $P_0|_{\eta=\eta^*}\neq 0$ if and only if $(M_N K_N L_N)^{(0,0)}$ is a $\mathbb{C}$-linear combination of $(M_N K_N L_N)^{(1,1)}$, …, $(M_N K_N L_N)^{(N,N)}$.

Proof:

There exist such P_k if and only if there exists a non-zero $\vec{v}_N$ in $\ker(D_\xi - A_N)\cap \ker(D_\eta - B_N)\cap\ker(M_N)$ such that $v_{i,j}|_{\xi=\xi^*}=0$ for all $0\le j < i \le N$. By Theorem VIII.6, this is equivalent to the existence of a non-zero $\vec{y}_N$ in $\ker(D_\xi) \cap \ker(D_\eta) \cap \ker(M_N K_N L_N)$ such that $(L_N \vec{y}_N)_{(i,j)} = (K_N L_N \vec{y}_N)_{(i,j)}|_{\xi=\xi^*} = 0$, for all $0 \leq j < i \leq N$. Notably $\vec{y}_N \in \mathbb{C}^{I_N}$ and gives a non-trivial linear relation on the entries of $M_N K_N L_N$. The presence of a $\vec{y}_N$ such that $y_{i,j}$ vanishes for $0 \leq j < i \leq N$ is equivalent to the presence a non-trivial linear relation on $(M_N K_N L_N)^{(0,0)}$, $(M_N K_N L_N)^{(1,1)}$, …, $(M_N K_N L_N)^{(N,N)}$. So completing the proof requires showing the $(L_N \vec{y}_N)_{(i,j)} = 0$ for $0 \leq j < i \leq N$ if and only if $y_{i,j} = 0$ for $0 \leq j < i \leq N$.

For any positive integer s, observe (8.13) gives that $D_\eta(D_\xi^j P_i)$, for $i - j \geq s$ depends only on $(D_\xi^{j'})P_{i'}$ for $i' - j' \geq s$. This gives a "graded" structure on this system of differential equations. Specifically, the equations giving $D_\eta(D_\xi^j P_i)$, for $i - j \geq s$, form a valid subsystem of differential equations. Thus $(L_N \vec{y}_N)_{(i,j)} = 0$ for $i - j \geq s$ if and only if $y_{i,j} = 0$ for $i - j \geq s$. Notably the case $s = 1$ gives us what we desire.

Note that $P_0|_{\eta=\eta^*} \neq 0$ is equivalent to $y_{0,0}$, which equals $v_{0,0}|_{\xi=\xi^*,\eta=\eta^*}$, being nonzero. In the context of the prior setting of this proof, $y_{0,0} \neq 0$ is equivalent to $(M_N K_N L_N)^{(0,0)}$ being a $\mathbb{C}$-linear combination of $(M_N K_N L_N)^{(1,1)}$, …, $(M_N K_N L_N)^{(N,N)}$. This yields the final portion of the theorem.

□

Coupling this with Theorem VII.4 gives a computational means for determining bounding within $\mathbb{C} \times \hat{\mathbb{C}}$ (for $N^- = 0$).

CHAPTER IX

A Calculational Scheme for a Characterization within $\mathbb{CP}^2$

Theorem IV.1, Theorem VI.2, and Theorem VI.6 have provided conditions equivalent to bounding within $\mathbb{CP}^2$. But these conditions do not appear immediately tractable. However in the case of $N^- = 0$ the previous chapter yielded results of a computable nature. In this chapter we establish some results which are favorable from a point of view of tractability and which make application of the previous chapter to the general bounding question.

Let γ be a closed, oriented, $\mathcal{C}^2$ real 1-chain. We call γ_1 a *sub-chain* of γ, if the components of γ_1 are contained in γ with the same orientation and no greater absolute multiplicity. If γ_1 is a sub-chain of γ, then it follows that $\gamma - \gamma_1$ is a sub-chain too. We say a collection of sub-chains *apportion* γ, if γ equals the sum of the sub-chains given.

From the previous chapter, recall the condition $(*_N)$ on μ, which was examined and shown equivalent to an integro-differential condition on μ through Theorem VIII.7. We may relate condition $(*_N)$ to the general bounding question through the following theorem.

Theorem IX.1. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \mathbb{CP}^2$, the following are equivalent:*

(i) γ bounds a holomorphic 1-chain within $\mathbb{CP}^2$

(v) $\exists$ (ξ^*, η^*) with some neighborhood Ω such that $\exists$ integers N^+ and N^- and two sub-chains γ^+ and γ^- apportioning γ such that

$$\mu^+(\xi, \eta) = \int_{\xi^*}^{\xi} \frac{\partial^2}{(\partial \xi')^2} G_{\gamma^+}(\xi', \eta)\, d\xi' \tag{9.1}$$

satisfies condition $(*_{N^+})$ and

$$\mu^-(\xi, \eta) = \int_{\xi^*}^{\xi} \frac{\partial^2}{(\partial \xi')^2} G_{-\gamma^-}(\xi', \eta)\, d\xi' \tag{9.2}$$

satisfies condition $(*_{N^-})$.

(v') Any (ξ^*, η^*) with any connected neighborhood $\Omega \subseteq \mathfrak{U}_\gamma$ satisfies (v).

Proof:

The implication (v') $\implies$ (v) trivially holds and the implication (v) $\implies$ (i) follows from the initial discussion of condition $(*_N)$ at the beginning of Chapter VIII.

So it remains to show that (i) $\implies$ (v'). So assume (i) and that γ bounds a holomorphic 1-chain V within $\mathbb{CP}^2$. Decompose V according to its positive and negative components to form V^+ and V^- such that $V = V^+ - V^-$ and V^+ and V^- locally contain no common components. Denote the boundary of V^+ as γ^+ and the boundary of $-V^-$ as γ^-. It is clear that $\gamma = \gamma^+ + \gamma^-$. If γ^+ and γ^- contain any common components with opposite orientation, then it would have to follow that V^+ and V^- locally share a common component. Thus any components common to γ^+ and γ^- must have the same orientation. It then follows γ^+ and γ^- are sub-chains that apportion γ.

Now γ^+ bounds the positive holomorphic 1-chain V^+ within $\mathbb{CP}^2$ and γ^- bounds the positive holomorphic 1-chain $-V^-$ within $\mathbb{CP}^2$. By Lemma VI.7, the first paragraph of its proof, and the discussion following the definition of condition $(*_N)$, (v') follows.

□

Any γ has only finitely many ways that it can be apportioned into two sub-chains γ^+ and γ^-. Corresponding μ^+ and μ^- may be explicitly determined for any choice of γ^+ and γ^-. So to test whether γ bounds a holomorphic 1-chain having prescribed bounds (N^+ and N^-) on its degrees of intersections with the perspective line amounts to testing whether any of finitely many pairs of μ^+ and μ^- satisfy condition $(*_{N^+})$ and condition $(*_{N^-})$, respectively.

Testing condition $(*_N)$ is tractable by Theorem VIII.7 and Theorem C.2. This implies that determining whether γ bounds a holomorphic 1-chain with prescribed bounds on the degrees of intersections with the perspective line may be achieved calculationally. Furthermore we note that satisfying condition $(*_N)$ is a closed condition. So bounding with some prescribed degree bounds also amounts to a closed condition. Therefore bounding (without degree bounds) constitutes an F_σ condition.

CHAPTER X

A New Approach and Alternate Characterizations within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$, $\mathbb{C} \times \hat{\mathbb{C}}$, and $\mathbb{C}^2$

One key item of the Dolbeault Henkin characterization is the form $\omega := z_1 \frac{dg}{g} = \frac{w_1}{w_0}\left(\frac{d(w_2-\xi w_0-\eta w_1)}{w_2-\xi w_0-\eta w_1} - \frac{dw_0}{w_0}\right)$, a meromorphic 1-form on $\mathbb{CP}^2$, parameterized by ξ and η. When integrated over γ, this form produces the function $G_\gamma(\xi, \eta)$ given in (4.1). We highlight two features of ω. One, when γ bounds a holomorphic 1-chain, the residues arising from ω can be well characterized. Two, the residues so produced contain enough information to fully describe a local portion of the holomorphic 1-chain bounded by γ. (A local portion of the holomorphic 1-chain then "seed" a global definition of a holomorphic 1-chain bounded by γ.) Boiling down to these two features, we consider this the philosophical essence of the characterization within $\mathbb{CP}^2$.

It is natural to consider if other forms could just as well serve in the place of ω. In this chapter we craft an alternate form. And with this form we produce parallel characterizations within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ (and so $\mathbb{CP}^2$ by birationality), $\hat{\mathbb{C}} \times \mathbb{C}$, and $\mathbb{C} \times \mathbb{C}$.

We explain the strategy for the construction of this new form. Assume we have a closed, oriented, C^2 real 1-chain in $\mathbb{C}^2$ which bounds a holomorphic 1-chain within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. (For $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ we will use coordinates $(z_0 : z_1) \times (w_0 : w_1)$ with associated affine coordinates (z, w) on $\mathbb{C}^2$. We define π_z and π_w as the corresponding projections onto

$\hat{\mathbb{C}}$.) If we slice V with the line $\{z = \zeta\}$ (with $\zeta \notin \pi_z(\gamma)$), then we get a holomorphic 0-chain in $\hat{\mathbb{C}}$ (unless V contains a component in $\{z = \zeta\}$), which has support over a discrete (and hence finite) set of points. Assuming there are no components of V in $\{w_0 = 0\}$, then for a generic choice of ζ, $\pi_w(V \cap \{z = \zeta\})$ resides in $\mathbb{C}$.

A finite divisor in $\mathbb{C}$ may be uniquely identified by its signed sum of positive integer powers (with multiplicity) and its total degree, meaning the total sum of multiplicities. (Since the total degree is the signed sum of zeroth powers, this is the same as using the sums of non-negative integer powers.) If we assume V lies in $\mathbb{C}^2$, then calculating the signed sum of mth powers of $\pi_w(V \cap \{z = \zeta\})$ may be achieved by integrating the form $\frac{1}{2\pi \mathrm{i}} w^m \frac{dz}{z-\zeta}$ over γ. Place these forms, for $m > 0$, in a generating function with respect to ξ. This generating function is

$$\sum_{i=0}^{\infty} w^{i+1} \frac{dz}{z-\zeta} \xi^i = \frac{w}{1 - w\xi} \frac{dz}{z-\zeta}. \tag{10.1}$$

The given series converges for appropriately small ξ and bounded w. It is useful to note that this form is the ξ-logarithmic derivative of the form $(\frac{1}{w} - \xi)^{-1} \frac{dz}{z-\zeta}$. (The *$\xi$-logarithmic derivative* of f is $(\log f)_\xi = \frac{f_\xi}{f}$.)

In agreeing with what appears to be standard combinatorial practice, we left the zeroth powers out of the generating function and consider the total degree in a separate manner. But one might peradventure use a generating function incorporating the zeroth powers. Ambiguity of the zeroth powers, or total degree, precisely correlates to ambiguity of the multiplicity of 0 in the divisor.

So define $\nu = \frac{w}{1-\xi w} \frac{dz}{z-\zeta} = \frac{w_1}{w_0 - \xi w_1} \left(\frac{z_0 dz_1 - z_1 dz_0}{z_0(z_1 - \zeta z_0)} \right)$, which is a (ξ, ζ) parameterized meromorphic 1-form in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Define H_γ as

$$H_\gamma(\zeta, \xi) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{w}{1 - \xi w} \frac{dz}{z - \zeta}, \tag{10.2}$$

which is the integration of ν over γ. Define $\mathfrak{U}_\gamma^\zeta = \mathbb{C} \backslash \pi_z(\gamma)$ and $\mathfrak{U}_\gamma^\xi = \mathbb{C} \backslash (1/\pi_w)(\gamma)$.

These sets have the same flavor as $\mathcal{U}_\gamma$ from Chapter IV, page 16. Also define $\hat{\mathcal{U}}^\zeta_\gamma = \hat{\mathbb{C}}\backslash\pi_z(\gamma)$. Note that $H_\gamma(\zeta,\xi)$ may be legitimately defined on $\hat{\mathcal{U}}^\zeta_\gamma \times \mathcal{U}^\xi_\gamma$, being 0 if $\zeta = \infty$. The following characterizations center on use of ν and H_γ, in a fashion analogous to ω and G_γ.

To facilitate these characterizations, we assume that π_z immerses $\operatorname{spt}\gamma$ into $\mathbb{C}$ with finitely many self-intersections. This doesn't appear to be an essential requirement, but it still remains to be explored how it may be best weakened or removed.

Theorem X.1. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ such that π_z gives an immersion of $\operatorname{spt}\gamma$ into $\mathbb{C}$ with finite self-intersections, the following are equivalent:*

1. *γ bounds a holomorphic 1-chain within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$.*
2. *For any (ζ^*,ξ^*) with any neighborhood Ω (coordinates (ζ,ξ)) in $\mathcal{U}^\zeta_\gamma \times \mathcal{U}^\xi_\gamma$ there exist functions $B(\zeta,\xi)$ and $C(\zeta,\xi)$ defined and meromorphic on Ω, with B being the ξ-logarithmic derivative of a function rational in ζ and C being the ξ-logarithmic derivative of a function rational in ξ, such that on Ω*

$$H_\gamma(\zeta,\xi) = B(\zeta,\xi) + C(\zeta,\xi). \tag{10.3}$$

3. *$\exists\ (\zeta^*,\xi^*)$ with a neighborhood Ω (coordinates (ζ,ξ)) such that there exists functions $B(\zeta,\xi)$ and $C(\zeta,\xi)$ defined and meromorphic on Ω with B being the ξ-logarithmic derivative of a function rational in ζ and C being the ξ-logarithmic derivative of a function rational in ξ, such that on Ω*

$$H_\gamma(\zeta,\xi) = B(\zeta,\xi) + C(\zeta,\xi). \tag{10.4}$$

Lemma X.2. *Let γ be a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ such that π_z gives an immersion of* $\operatorname{spt}\gamma$ *into $\mathbb{C}$ with finite self-intersections. Suppose γ bounds a holomorphic 1-chain V within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ with no components in $\mathbb{V}(z_0 w_0)$. On $\mathcal{U}^\zeta_\gamma \times \mathcal{U}^\xi_\gamma$, there exists meromorphic functions $B(\zeta,\xi)$ and $C(\zeta,\xi)$ with B being the ξ-logarithmic derivative of a function rational in ζ and C being the ξ-logarithmic derivative of a function rational in ξ, such that $H_\gamma(\zeta,\xi) = B(\zeta,\xi) + C(\zeta,\xi)$.*

Proof (of Lemma): Define $\mathcal{T}^\zeta_V = \{\zeta \in \mathcal{U}^\zeta_\gamma \mid \{z_1 = \zeta z_0\}$ is not locally transverse to $V\}$ and $\mathcal{T}^\xi_V = \{\xi \in \mathcal{U}^\xi_\gamma \mid \{w_0 = \xi w_1\}$ is not locally transverse to $V\}$. Define $\mathcal{I}^\zeta_V = \{\zeta \in \mathcal{U}^\zeta_\gamma \mid \{z_1 = \zeta z_0\} \cap V \cap \{w_0 = 0\} \neq \emptyset\}$ and $\mathcal{I}^\xi_V = \{\xi \in \mathcal{U}^\xi_\gamma \mid \{w_0 = \xi w_1\} \cap V \cap \{z_0 = 0\} \neq \emptyset\}$.

Let $(\zeta^*,\xi^*) \in (\mathcal{U}^\zeta_\gamma \backslash \mathcal{T}^\zeta_V) \times (\mathcal{U}^\xi_\gamma \backslash \mathcal{T}^\xi_V)$. For z near ζ^*, V may be suitably represented as $\sum_j \epsilon_j \mathbb{V}(w - w_j(z))$, where $w_j(z)$ is a meromorphic function for z near ζ^*, and ϵ_j gives an integer multiplicity. For w near $\frac{1}{\xi^*}$, V can be represented as $\sum_j \epsilon_j \mathbb{V}(z - z_j(1/w))$, where $z_j(1/w)$ is a meromorphic function for $1/w$ near ξ^*.

For ζ near $\zeta^* \in \mathcal{U}^\zeta_\gamma \backslash \mathcal{T}^\zeta_V$ and $\xi \in \mathcal{U}^\xi_\gamma$, we define

$$C(\zeta,\xi) = \sum_j \epsilon_j \left. \frac{w_j(z)}{1 - w_j(z)\xi} \right|_{z=\zeta}. \tag{10.5}$$

Note that this is the ξ-logarithmic derivative of $\prod_j (1 - w_j(\zeta)\xi)^{-\epsilon_j}$. By symmetry and a removable singularities argument, we can meromorphically extend this to a single-valued definition on $(\zeta,\xi) \in \hat{\mathcal{U}}^\zeta_\gamma \times \mathcal{U}^\xi_\gamma$, which is a ξ-logarithmic derivative of a function rational in ξ.

For ξ near $\xi^* \in \mathcal{U}^\xi_\gamma \backslash \mathcal{T}^\xi_V$ and $\zeta \in \mathcal{U}^\zeta_\gamma$, we define

$$B(\zeta,\xi) = \sum_j \epsilon_j \left. \frac{z_j'(1/w)}{z_j(1/w) - \zeta} \right|_{1/w=\xi} - \sum_j \epsilon_j \left. \frac{w_j(z)}{1 - w_j(z)\xi} \right|_{z=\infty}, \tag{10.6}$$

where the second term may also be denoted $-C(\infty, \xi)$, and well-defined by the previous. $B(\zeta, \xi)$ is the ξ-logarithmic derivative of $\prod_j (z_j(\xi) - \zeta)^{\epsilon_j} \prod_j (1 - w_j(\infty)\xi)^{\epsilon_j}$. Again by symmetry and removing singularities, we can meromorphically extend this to a single-valued definition on $(\zeta, \xi) \in \hat{\mathcal{U}}_\gamma^\zeta \times \mathcal{U}_\gamma^\xi$. This gives $B(\zeta, \xi)$ as being the ξ-logarithmic derivative of a function rational in ζ.

Assuming that V is locally transverse to $\{z_0 = 0\}$ and $(\zeta, \xi) \in (\mathcal{U}_\gamma^\zeta \backslash \mathcal{T}_V^\zeta) \times (\mathcal{U}_\gamma^\xi \backslash \mathcal{T}_V^\xi)$, then $H_\gamma(\zeta, \xi) = B(\zeta, \xi) + C(\zeta, \xi)$ holds by elementary residue calculations. By extension, this holds for $(\zeta, \xi) \in \mathcal{U}_\gamma^\zeta \times \mathcal{U}_\gamma^\xi$. By taking the limit of "horizontal" perturbations of V, we may drop the assumption that V be locally transverse to $\{z_0 = 0\}$.

□

Remark: We could alternatively move the right-most term of the definition for $B(\zeta, \xi)$ to the definition for $C(\zeta, \xi)$. The proof would still remain valid as the term to be transferred is the ξ-logarithmic derivative of a function rational in ξ and constant in ζ. However $C(\zeta, \xi)$, as defined by (10.5), is the generating function of sums of powers of the w-coordinates of $V \cap \{z = \zeta\}$. This may be seen by the following equation.

(10.7)
$$\sum_j \epsilon_j \left. \frac{w_j(z)}{1 - w_j(z)\xi} \right|_{z=\zeta} = \sum_j \epsilon_j \left. \sum_{k=0}^{\infty} w_j(z)^{k+1} \xi^k \right|_{z=\zeta} = \sum_{k=0}^{\infty} \left(\sum_j \epsilon_j w_j(\zeta)^{k+1} \right) \xi^k.$$

Now we present some lemmas in regard to rationality.

Lemma X.3. *Let R be a Laurent series of the form $\sum_{q=-\infty}^{N} \sigma_q t^j$, $N \in \mathbb{Z}$. R gives a rational function in t if and only if for some $\ell > 0$, $\det[\Sigma_{i_0}\ \Sigma_{i_1} \cdots \Sigma_{i_\ell}] = 0$ for all $0 > i_0 > i_1 > \cdots > i_\ell$, where $\Sigma_i^{\mathrm{T}} = [\sigma_i\ \sigma_{i-1} \cdots \sigma_{i-\ell+1}]$.*

This lemma is routinely present in the literature on boundaries of holomorphic chains. This statement and proof are essentially quoted from [17], where it is attributed as classical work due to Hadamard.

Proof: Note the following sequence of equivalent statements.

1. R represents a rational function with a denominator of degree at most ℓ.
2. $\exists$ polynomials $Q = \sum_{j=0}^{\ell} d_j t^j$ and P such that $QR = P$.
3. $\exists$ $(d_0, d_1, \ldots, d_\ell) \in \mathbb{C}^{\ell+1}$ such that $\sum_{j=0}^{\ell} d_j \sigma_{q-j} = 0$, for all $q < 0$.
4. $\exists$ $\ell > 0$, $\det[\Sigma_{i_0}\ \Sigma_{i_1} \cdots \Sigma_{i_\ell}] = 0$ for all $0 > i_0 > i_1 > \cdots > i_\ell$, where $\Sigma_i^{\mathrm{T}} = [\sigma_i\ \sigma_{i-1} \cdots \sigma_{i-\ell+1}]$.

□

Lemma X.4. *Let Ω be a domain in $\mathbb{C}$ with an arc α in the boundary. Suppose that $f_j(z)$ are functions continuous on $\Omega \cup \alpha$ and holomorphic on Ω, such that $f(z, w) = \sum_{j=\infty}^{N} f_j(z) w^j$ on $(\Omega \cup \alpha) \times U$, for some open set U not containing zero. If $f(z, w)$ is rational in w with a denominator of degree at most ℓ for $z \in \alpha$, then $f(z, w)$ is rational in w with a denominator of degree at most ℓ for $z \in \Omega$.*

Proof: $\det[\Sigma_{i_0}(z)\ \Sigma_{i_1}(z) \cdots \Sigma_{i_\ell}(z)]$ for $0 > i_0 > i_1 > \cdots > i_\ell$, where $\Sigma_i^{\mathrm{T}}(z) = [f_i(z)\ f_{i-1}(z) \cdots f_{i-\ell+1}(z)]$, are functions continuous on $\Omega \cup \alpha$ and holomorphic on Ω. By Lemma X.3 these vanish for $z \in \alpha$, thus they vanish on Ω, and by the reverse application of Lemma X.3, this lemma holds.

□

We also will employ a result of Plemelj, which we simply state here and refer to [21].

Lemma X.5. *Let α be a simple $\mathcal{C}^1$ arc. Let ϕ be a Hölder continuous function on α. Let U^+ and U^- be two disjoint domains containing α in the boundary. Assume that α is oriented positively with respect to U^+ and negatively with respect to U^-. The function Φ^+ on U^+, given by*

$$\Phi^+(z_0) = \frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{\phi(z)\, dz}{z - z_0} \tag{10.8}$$

has a continuous extension to $U^+ \cup \alpha$ and the function Φ^- on U^-, given by

$$\Phi^-(z_0) = \frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{\phi(z)\, dz}{z - z_0} \tag{10.9}$$

has a continuous extension to $U^- \cup \alpha$. These continuations also satisfy

$$\Phi^+(z_0) - \Phi^-(z_0) = \phi(z_0) \tag{10.10}$$

for $z_0 \in \alpha$, which we refer to as a jump condition.

Proof (of Theorem): Note that $2 \implies 3$ is trivial and that Lemma X.2 establishes $1 \implies 2$. So it remains to show $3 \implies 1$.

Choose U' and U neighborhoods of ζ^* and ξ^*, respectively, such that $U' \times U \subseteq \Omega$. Then by the condition on B there exist non-negative integers M and N and functions $p_0(\xi), p_1(\xi), \ldots, p_M(\xi)$ and $q_0(\xi), q_1(\xi), \ldots, q_N(\xi)$ analytic on U such that

(10.11)
$$B(\zeta, \xi) = \frac{p_0'(\xi) + p_1'(\xi)\zeta + \cdots + p_M'(\xi)\zeta^M}{p_0(\xi) + p_1(\xi)\zeta + \cdots + p_M(\xi)\zeta^M} - \frac{q_0'(\xi) + q_1'(\xi)\zeta + \cdots + q_M'(\xi)\zeta^N}{q_0(\xi) + q_1(\xi)\zeta + \cdots + q_M(\xi)\zeta^N}$$

on $U' \times U$. With this definition we can extend B meromorphically to $\hat{\mathbb{C}} \times U$. For later use, note that the pole set of B contains no components in any variety given as $\zeta = \zeta_0$. For if (ζ_0, ξ) were a pole of B for all $\xi \in U$, then ζ_0 must be a root (of some degree) of one the denominators of (10.11). But then it would be a root (of the same degree or more) of the corresponding numerator.

Define

$$G(\zeta,\xi) = H_\gamma(\zeta,\xi) - B(\zeta,\xi) \tag{10.12}$$

meromorphically on $\mathfrak{U}^\zeta_\gamma \times U$. For $\lambda \in U$, define $P_\lambda = \{\zeta \in \mathfrak{U}^\zeta_\gamma \mid G(\zeta,\lambda) = \infty\}$. Since $H_\gamma(\zeta,\xi)$ is holomorphic on this domain, note that the pole sets of G and B are equivalent. In particular the pole set of G does not contain components in lines of the form $\{\zeta = \zeta_0\}$. Therefore P_λ is always a discrete set.

Also choose an integer b_0 and define $\kappa_0(\zeta) = \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{dz}{z-\zeta} - b_0$. This is locally constant and defined on $(\hat{\mathbb{C}}\backslash\pi_z(\gamma)) \times U$.

Motivation: The function G is an extension of the function C. We will view G as prescribing the generating function of signed sums of powers of the w-coordinates of a holomorphic 1-chain sliced with $\{z = \zeta\}$. Along with $\kappa_0(\zeta)$, which will prescribe the total degree on such slices, we maintain this provides enough information to (uniquely) construct a holomorphic 1-chain bounded by γ. However this requires G to be of a particular form for it to encode "valid data", namely to be the ξ-logarithmic derivative of a function rational in ξ. So the next section of this proof is devoted to showing this particular form holding for C must continue to G. Our method to do this is to look at the "ξ-logarithmic integral" of $-G(\zeta,\xi)$. In fact the logarithmic integral may be used to give a function whose divisor is locally the holomorphic 1-chain we wish to construct. To illustrate this point, consider $C(\zeta,\xi)$ in the form of (10.5), with $c_0 = \sum_j \epsilon_j$ denoting the total degree. Then for $(z,w) \in U' \times (1/U)$,

$$w^{c_0} \exp\left(-\int_\lambda^{\frac{1}{w}} C(z,\xi)\, d\xi\right) = \prod_j \left(\frac{w - w_j(z)}{1 - w_j(z)\lambda}\right)^{\epsilon_j}. \tag{10.13}$$

This outlines the path remaining for this proof.

For $\lambda \in U$ define $Q_\lambda = \{(z,w) \in \mathfrak{U}^\zeta_\gamma \times (1/U) \mid z \notin P_\lambda \cup P_{\frac{1}{w}}\} = \{(z,w) \in \mathfrak{U}^\zeta_\gamma \times (1/U) \mid G(z,\lambda) \neq \infty, G(z,\frac{1}{w}) \neq \infty\}$. On Q_λ, define the non-vanishing holomorphic function F_λ as

$$F_\lambda(z,w) = w^{\kappa_0(z)} \exp\left(-\int_\lambda^{\frac{1}{w}} G(z,\xi)\, d\xi \right). \tag{10.14}$$

Recall (10.12) and note that on $\mathfrak{U}_\gamma^\zeta \times U$, H is a holomorphic function and B is the ξ-logarithmic derivative of a meromorphic function. $B(\zeta,\xi)$ may be represented as $\frac{R_\xi(\zeta,\xi)}{R(\zeta,\xi)}$, where $R(\zeta,\xi)$ is a meromorphic function. The set of poles and zeros for R is equal to the pole set for B and thus the pole set for G. Now

$$-\int_\lambda^{\frac{1}{w}} G(z,\xi)\, d\xi = -\int_\lambda^{\frac{1}{w}} H_\gamma(z,\xi)\, d\xi - \log(R(z,\frac{1}{w})) + \log(R(z,\lambda)) \tag{10.15}$$

Note this function is holomorphic on Q_λ. Though it is technically multi-valued, it is single-valued modulo addition of integer multiples of 2πi. For one this shows that (10.14) constitutes a valid definition for F_λ. By analyticity, locally about points in Q_λ, the expression in (10.15) has a series expansion of the form

$$\sum_{j=0}^{\infty} a_j(z)(\frac{1}{w})^j, \tag{10.16}$$

where $a_j(z)$ are locally defined holomorphic functions. We can similarly derive such series for $\exp\left(-\int_\lambda^{\frac{1}{w}} G(z,\xi)\, d\xi\right)$. Thus we have a Laurent series for F_λ of the form

$$F_\lambda(z,w) = \sum_{j=-\infty}^{\kappa} f_{\lambda,j}(z) w^j, \tag{10.17}$$

where $f_{\lambda,j}(z)$ are locally defined holomorphic functions.

Furthermore, on $Q_{\lambda_1} \cap Q_{\lambda_2}$, the expression

$$\frac{F_{\lambda_2}(z,w)}{F_{\lambda_1}(z,w)} = \exp\left(-\int_{\lambda_2}^{\lambda_1} G(z,\xi)\, d\xi \right) \tag{10.18}$$

is independent of w. Observe F_{λ_1} is rational with respect to w if and only if F_{λ_2} is.

For $(\zeta,\xi) \in U' \times U$, $G(\zeta,\xi) = C(\zeta,\xi)$ is the ξ-logarithmic derivative of a function rational in ξ, by the theorem's assumption. Therefore for all $\lambda \in U$, F_λ is rational with respect to w on $Q_\lambda \cap (U' \times 1/U)$.

Now consider λ fixed. For polydiscs contained compactly within Q_λ, we may apply Lemma X.4 and see that rationality in w of F_λ continues within connected components of Q_λ. We will get rationality in w of F_λ on all of Q_λ by showing rationality in w transfers over arcs in $\pi_z(\gamma)$.

Let α be a simple arc in $\pi_z(\gamma)$ avoiding the self-intersection set and $\overline{P_\lambda}$ and separating two connected components of $\mathbb{C}\backslash\pi_z(\gamma)$, which we'll label U^+ and U^-. Assume α is oriented positively with respect to U^+ and negatively with respect to U^-. (In other words, local to α, $d[U^+] = [\alpha] = -d[U^-]$.) We suppose that F_λ is rational in w over U^- (technically $Q_\lambda \cap (U^- \times (1/U))$. Define m to be the multiplicity (possibly negative) of α in $\pi_z(\gamma)$. (In application we may reverse the direction of α and the sign of m if we have F_λ being rational in w over U^+.)

Because of the immersion assumption on $\operatorname{spt}\gamma$, we may let γ be given over α as $z \in \alpha \mapsto (z, f(z))$, for some function f.

By Plemelj's Theorem, given as Lemma X.5, H_γ continuously extends to α from either side. Namely there exist functions H_γ^+ continuous on $U^+ \cup \alpha$ and H_γ^- continuous on $U^- \cup \alpha$ that agree with H_γ on U^+ and U^-, respectively. Plemelj's Theorem also describes the jump condition between these two continuous extensions to α. This yields that $H_\gamma^+(\zeta,\xi) - H_\gamma^-(\zeta,\xi) = m\frac{f(\zeta)}{1-\xi f(\zeta)}$, for $\zeta \in \alpha$. We get similar extensions κ_0^+ and κ_0^- of κ_0 with the jump condition $\kappa_0^+ - \kappa_0^- = m$. These continuous extensions and determinations of the jump conditions then cascade through the definitions for G and F_λ. Notably the jump condition, $G^+ - G^-$, for G is the same as that for $H_\gamma^+ - H_\gamma^-$. This gives a multiplicative jump condition on F_λ.

$$\begin{aligned}
\text{(10.19)}\quad \frac{F_\lambda^+(z,w)}{F_\lambda^-(z,w)} &= w^{\kappa_0^+(z)-\kappa_0^-(z)} \exp\left(-\int_\lambda^{\frac{1}{w}} G^+(z,\xi) - G^-(z,\xi)\, d\xi \right) \\
&= w^m \exp\left(m \int_\lambda^{\frac{1}{w}} \frac{-f(z)}{1-\xi f(z)}\, d\xi \right) = w^m \left(\frac{1-\frac{1}{w}f(z)}{1-\lambda f(z)} \right)^m = \left(\frac{w-f(z)}{1-\lambda f(z)} \right)^m
\end{aligned}$$

For $z \in \alpha$, there will exist a small neighborhood about it, U''', in either $U^+ \cup \alpha$ or $U^- \cup \alpha$ and a open set U'' in $(1/U)$ such that $(U'''\backslash\alpha) \times U'' \in Q_\lambda$, on which we get a Laurent series as assumed for Lemma X.4. Therefore rationality in w holds for F_λ^- for $z \in \alpha$. By the multiplicative jump condition this carries over to F_λ^+ for $z \in \alpha$, though possibly with a denominator of higher degree. This then implies rationality in w of F_λ for z over U^+.

$\mathbb{C}\backslash\pi_z(\gamma)$ consists of only finitely many components. Therefore by the previous, F_λ is rational with respect to w on Q_λ, and may be extended meromorphically (rationally in w) to $(\mathfrak{U}_\gamma^\zeta\backslash P_\lambda) \times \hat{\mathbb{C}}$. Define S_λ to be the divisor of F_λ on $(\mathfrak{U}_\gamma^\zeta\backslash P_\lambda) \times \hat{\mathbb{C}}$. As (10.18) gives that F_{λ_1} multiplicatively differs from F_{λ_2} by a non-vanishing holomorphic function independent of w, S_{λ_1} and S_{λ_2} agree on $(\mathfrak{U}_\gamma^\zeta\backslash(P_{\lambda_1} \cup P_{\lambda_2})) \times \hat{\mathbb{C}}$. So we may glue the S_λ together to form a holomorphic 1-chain in $\mathfrak{U}_\gamma^\zeta \times \hat{\mathbb{C}}$. By the jump conditions and some arguments employed in Harvey and Lawson [15], we can extend S to $(\hat{\mathbb{C}} \times \hat{\mathbb{C}})\backslash \operatorname{spt}\gamma$ and note that $d[S] = [\gamma]$.

$\square$

From this we can derive another characterization for $\mathbb{C} \times \hat{\mathbb{C}}$

Theorem X.6. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \mathbb{C} \times \hat{\mathbb{C}}$ such that π_z gives an immersion of* $\operatorname{spt}\gamma$ *into $\mathbb{C}$ with finite self-intersections, the following are equivalent:*

1. *γ bounds a holomorphic 1-chain within $\mathbb{C} \times \hat{\mathbb{C}}$*
2. *For any (ζ^*, ξ^*) with any connected neighborhood Ω (coordinates (ζ, ξ)) in $\mathfrak{U}_\gamma^\zeta \times \mathfrak{U}_\gamma^\xi$ with ζ^* in the unbounded component of $\mathfrak{U}_\gamma^\zeta$, there exists a function B defined and analytic on Ω and being the ξ-logarithmic derivative of a function rational*

in ζ, such that on Ω

$$H_\gamma(\zeta,\xi) = B(\zeta,\xi). \tag{10.20}$$

3. *$\exists$ (ζ^*,ξ^*) with a connected neighborhood Ω (coordinates (ζ,ξ)) with ζ^* in the unbounded component of $\mathfrak{U}_\gamma^\zeta$, such that there exists a function B defined and analytic on Ω and being the ξ-logarithmic derivative of a function rational in ζ, such that on Ω*

$$H_\gamma(\zeta,\xi) = B(\zeta,\xi). \tag{10.21}$$

Proof: Let U_0' denote the unbounded component of $\mathbb{C}\backslash\pi_z(\gamma)$. Assume condition 1, that is there is a holomorphic 1-chain V bounded by γ within $\mathbb{C}\times\hat{\mathbb{C}}$. By the maximum principle it holds that $\operatorname{spt} V \cap U_0' \times \hat{\mathbb{C}} = \emptyset$. Therefore proceeding via the calculation in Lemma X.2 with $\Omega \subset U_0 \times \hat{\mathbb{C}}$, we get that $C(\zeta,\xi) = 0$ on Ω. Thus 1 $\implies$ 2.

2 clearly implies 3, so it only remains to show the 3 implies 2. So assume condition 3 which also implies condition 3 of Theorem X.1, using B as given and C given as being zero. Applying the proof of Theorem X.1, one constructs a holomorphic 1-chain V bounded by γ. Using $C = 0$ in Ω, along with choosing b_0 such that $\kappa_0(\zeta^*) = 0$, gives that F_λ equals 1 for z near ζ^*. Thus the constructed holomorphic 1-chain V does not intersect the line $\{z = \zeta^*\}$. This implies $\operatorname{spt} V \subset (\mathbb{C}\backslash U_0) \times \hat{\mathbb{C}}$. So V is bounded by γ within $\mathbb{C}\times\hat{\mathbb{C}}$.

$\square$

A slightly different statement along similar grounds is the following.

Theorem X.7. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2 \subset \hat{\mathbb{C}} \times \mathbb{C}$ such that π_z gives an immersion of $\operatorname{spt}\gamma$ into $\mathbb{C}$ with finite self-intersections, the following are equivalent:*

1. *γ bounds a holomorphic 1-chain within $\hat{\mathbb{C}} \times \mathbb{C}$*

2. *For any (ζ^*, ξ^*) with any neighborhood Ω (coordinates (ζ, ξ)) in $\mathfrak{U}^\zeta_\gamma \times \mathfrak{U}^\xi_\gamma$ with ξ^* in the component of $\mathfrak{U}^\xi_\gamma$ containing zero, there exists a function C defined and analytic on Ω and being the ξ-logarithmic derivative of a function rational in ξ, such that on Ω*

$$H_\gamma(\zeta, \xi) = C(\zeta, \xi). \tag{10.22}$$

3. *$\exists$ (ζ^*, ξ^*) with a neighborhood Ω (coordinates (ζ, ξ)) with ξ^* in the component of $\mathfrak{U}^\xi_\gamma$ containing zero, such that there exists a function C defined and analytic on Ω and being the ξ-logarithmic derivative of a function rational in ξ, such that on Ω*

$$H_\gamma(\zeta, \xi) = C(\zeta, \xi). \tag{10.23}$$

Proof: Let U_0 be the component of $\mathfrak{U}^\xi_\gamma$ containing 0. Assume 1, so let V be a holomorphic 1-chain bounded by γ within $\hat{\mathbb{C}} \times \mathbb{C}$. Apply Lemma X.2, with the modification that the second term for the expression for B in (10.6) be instead transferred to the expression for C in (10.5). (See the remark following that lemma.) The B so derived will be zero, thus implying 2.

Clearly 2 $\implies$ 3. Now assume 3. We use the construction from the proof of 3 $\implies$ 1 of Theorem X.1, using $B = 0$ and C as given. Thus $G(\zeta, \xi)$ is holomorphic on $\mathfrak{U}^\zeta_\gamma \times U$, implying $P_\lambda = \emptyset$ and thus F_λ is holomorphic and non-vanishing on $\mathfrak{U}^\zeta_\gamma \times (1/U)$.

As $\xi^* \in U$, this gives that the constructed holomorphic 1-chain V (which is bounded by γ) does not intersect $\{w_0 = \xi^* w_1\}$. Therefore $\operatorname{spt} V \in \hat{\mathbb{C}} \times (\mathbb{C} \backslash (1/U_0))$. Thus 1 follows.

□

And in the same vein, we can derive an analogous result for $\mathbb{C}^2$.

Theorem X.8. *For γ a closed, oriented, $\mathcal{C}^2$ real 1-chain in $\mathbb{C}^2$ such that π_z gives an immersion of* $\operatorname{spt}\gamma$ *into $\mathbb{C}$ with finite self-intersections, the following are equivalent:*

1. *γ bounds a holomorphic 1-chain within $\mathbb{C}^2$*
2. *For any (ζ^*, ξ^*) with any neighborhood Ω (coordinates (ζ, ξ)) in $\mathfrak{U}^{\zeta}_{\gamma} \times \mathfrak{U}^{\xi}_{\gamma}$ with ζ^* in the unbounded component of $\mathfrak{U}^{\zeta}_{\gamma}$ and ξ^* in the component of $\mathfrak{U}^{\xi}_{\gamma}$ containing zero, then on Ω*

$$H(\zeta, \xi) = 0. \tag{10.24}$$

3. *$\exists$ (ζ^*, ξ^*) with a neighborhood Ω (coordinates (ζ, ξ)) with ζ^* in the unbounded component of $\mathfrak{U}^{\zeta}_{\gamma}$ and ξ^* in the component of $\mathfrak{U}^{\xi}_{\gamma}$ containing zero, then on Ω*

$$H(\zeta, \xi) = 0. \tag{10.25}$$

Proof: Let U_0' be the unbounded component of $\mathfrak{U}^{\zeta}_{\gamma}$ and U_0 be the component of $\mathfrak{U}^{\xi}_{\gamma}$ containing zero. $1 \implies 2$ follows by application of the proof Lemma X.2. $2 \implies 3$ follows trivially.

Now assume 3 and apply the construction from the proof of Theorem X.1 to construct a holomorphic 1-chain V bounded by γ within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$, using B and C as zero in the given neighborhood Ω of (ζ^*, ξ^*). Examining the procedure (it is helpful to note the proofs of Theorem X.6 and Theorem X.7) shows that the constructed holomorphic 1-chain V has support contained in $(\mathbb{C}\backslash U_0') \times (\mathbb{C}\backslash(1/U_0))$. Thus 1 holds.

$\square$

The system of moment conditions for γ in $\mathbb{C}^2$ is equivalent to the following reduced system of monomial moment conditions on γ.

(10.26)
$$\int_\gamma w^m z^n \, dz = 0 \text{ for all } m \geq 1,\, n \geq 0.$$

Now observe that $\frac{w}{1-w\xi} = \sum_{i=0}^{\infty} w^{i+1}\xi^i$ and $\frac{dz}{z-\zeta} = \sum_{i=0}^{\infty} z^i \zeta^{-(i+1)} dz$. So the monomial moments $\int_\gamma w^m z^n \, dz$ exactly constitute the coefficients in the Taylor expansion of $H(\zeta, \xi)$ about $(\infty, 0)$. This gives an elegant correlation between Theorem X.8 and the characterization within $\mathbb{C}^2$ due to moment conditions.

CHAPTER XI

Conclusion

Thus far we have examined a number of issues concerning characterizations within the surfaces $\mathbb{CP}^2$ (or its birational equivalent $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$), $\mathbb{C} \times \hat{\mathbb{C}}$, and $\mathbb{C}^2$. We have given some adaptations for making the Dolbeault and Henkin characterization within $\mathbb{CP}^2$ more versatile and tractable. Also we demonstrated that for a generic set of closed, oriented, $\mathcal{C}^2$ real 1-chains that being a boundary of a holomorphic 1-chain is invariant under birational maps. We employed this in forming the first characterization within $\mathbb{C} \times \hat{\mathbb{C}}$. Furthermore, by developing a distinctly different approach, we demonstrated additional characterizations for $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$, $\mathbb{C} \times \hat{\mathbb{C}}$, and even $\mathbb{C}^2$.

There remain many questions and avenues for continued inquiry and research. Presently, we possess two characterizations for $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$, one derived from the form ω and one from ν. This similarly holds for $\mathbb{C} \times \hat{\mathbb{C}}$. We may refer to these as the ω and ν characterizations, respectively. The presence of these two families of characterizations raises the following three questions.

First, how do these characterizations compare and contrast? We have not presented a means of calculation for the ν characterizations, but that would be one element for comparison. The rationality test due to Hadamard, Lemma X.3 should give a calculational means for determining boundaries of holomorphic 1-chains within

$\mathbb{C} \times \hat{\mathbb{C}}$. However a calculational means for boundaries of holomorphic 1-chains within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ appears to require more intense study. Also there are certain refinements to the computational method for ω characterizations that appear feasible though not yet confirmed. For instance instead of having condition $(*_N)$ being expressed as an integro-differential equation condition, it appears it could possibly be expressed purely as a differential equation condition. (Already the author can remove the need for anti-differentiation in η. Doing the same for ξ appears promising, but needs further examination.)

Secondly, what forms, other than ω and ν, could be used to produce characterizations within these surfaces? The introduction to Chapter X describes what appears to be the essential factors for a form to generate a characterization. So one aim is to produce a general class of forms with corresponding characterizations.

Thirdly, can we produce such forms, and resulting characterizations, on other complex surfaces? In this regard, one promising class of targets is the products of Riemann surfaces. A suitable selection of functions that separate points on a complex curve then seems to direct the construction a generating function that separates divisors. One may observe this in the philosophy of selecting the generating function $\frac{w}{1-\xi w}$ in Chapter X. One immediate focus would be on products of Riemann surfaces. Other potential spaces of study include line bundles and projective bundles over Riemann surfaces plus Zariski-open subsets of rational complex surfaces.

Separate from the previous sequence of questions there are others to be explored. One topic for further research is looking into the use of birational maps in porting characterizations to other surfaces. Also we may wish to broaden our scope beyond the case of dimension one and codimension one. There is some promise in generalizing the methods of Chapter X to $\mathbb{C}^r \times \hat{\mathbb{C}}^s$.

Reaching beyond just those questions which seem immediately accessible, there appear some rather grand, inspiring questions. What do the boundaries of holomorphic chains communicate about an ambient space? Is there a uniform rubric or template for characterizations so to facilitate comparisons of ambient spaces? What suitable classifications of complex spaces are possible by viewing the collection the boundaries of holomorphic chains a space contains? For instance, Stein spaces appear logically as one class. But amongst the non-Stein spaces, what distinguishing properties are natural from this point of view?

The work and results developed in this dissertation contribute to the substance and motivation in studying the boundaries of holomorphic chains. These concluding questions demonstrate that the area is fertile and has ample promise for further fruit.

APPENDICES

APPENDIX A

A Classical Model Relevant to Studying the Behavior of Holomorphic 1-Chains near a Line

In considering the behavior of a variety near a line in $\mathbb{CP}^2$, we consider a particularly simple yet general model in this section.

We use coordinates (x, y) for $\mathbb{C}^2$. An analytic variety in $\mathbb{C}\times\hat{\mathbb{C}}$ with no components of the form $\{x = x_0\}$ can be viewed as a branched cover of $\mathbb{C}$, with the projection map $\pi(x, y) = x$. If x_0 is not a branch point and the variety is bounded near $x = x_0$, then the variety near $x = x_0$ may be locally given as the graphs of holomorphic functions $y = f_j(x)$. In this case the f_j have a convergent Taylor expansion of the form

$$f(x) = \sum_{j=0}^{\infty} c_j (x - x_0)^j. \tag{A.1}$$

If the variety should be unbounded near $x = x_0$, then a Laurent expansion would be required with meromorphic f_j. If x_0 were a branch point, then a Puiseux expansion (permitting fractional exponents) would be required to express the multi-valued f_j. By a multi-valued function, we mean a global analytic function in the sense of Weierstrass. (Ahlfors [2](Chapter 8 Sections 1 and 2) provides some appropriate definitions followed by a study of the above in the case of an algebraic variety.)

We confine our focus to a local region about the line $x = 0$. Let $\Delta_r = \{x \mid |x| < r\}$, $\Delta_r^* = \{x \mid 0 < |x| < r\}$ and $S_r^{\theta_1,\theta_2} = \{x \mid 0 < |x| < r, \exists n \text{ s.t. } \theta_1 < \arg(x)+n2\pi < \theta_2\}$.

We define a Laurent-Puiseux germ (about 0) f as a holomorphic function defined, for some small r, on an open sector $S_r^{\theta_1,\theta_2}$ intersecting the positive real axis (i.e. $\theta_1 < 0 < \theta_2$) such that analytic continuation counter-clockwise about the origin h times produces the original function on $S_r^{\theta_1,\theta_2}$ for some positive h. Equivalently, it may be defined as a single valued function on a h sheeted cover of the small punctured disc Δ_r^*. If h is the smallest non-zero number such that this occurs, then we say f is h-valued. For our purposes here, we will further assume that a Laurent-Puiseux germ is locally meromorphic at $x = 0$. This means that f is $O(|x|^m)$ as $x \to 0$ for some $m \in \mathbb{Z}$.

(What we call a Laurent-Puiseux germ might possibly be more appropriately called a multi-valued meromorphic germ. However this is presuming finite-sheeted as part of our meaning for multi-valued, which may or may not be standard. But in any case, for conciseness of notation, we use Laurent-Puiseux germs or multi-valued meromorphic germs synonymously to mean the definition given above, which is admittedly tailored for our purposes here.)

Let f be a Laurent-Puiseux germ. We term the kth *associate* of f to be the Laurent-Puiseux germ produced by analytic continuation about the origin k times counter-clockwise, starting with f. We define the *graph* of f to be the graph of f and all its continuations. Note that f has the same graph as its associates.

For a Laurent-Puiseux germ, there is a Laurent-Puiseux expansion about 0 of the form

$$f(x) = \sum_{j=-M}^{\infty} c_{j/h} x^{j/h} = \sum_{j=-M}^{\infty} c_{j/h} \xi^j, \tag{A.2}$$

where we understand ξ as $x^{1/h}$, viewed as a Laurent-Puiseux germ or as the coordinate for a h sheeted cover of Δ_r^*. The Laurent-Puiseux expansion of the kth associate is

$$\sum_{j=-M}^{\infty} c_{j/h}\omega^{kj}x^{j/h}, \tag{A.3}$$

where ω is the hth primitive root of unity. We may typically refer to the Laurent-Puiseux expansion of f as

$$f(x) = \sum_p c_\alpha x^p, \tag{A.4}$$

where we understand the c_p to zero for p off of some subset of a finitely generated rational lattice bounded from below. (In this section we may notate a sum over an infinite index set, being considered valid so long as the indices with non-zero terms are finite or discrete and so prescribe a legitimate sum or series.)

The set of Laurent-Puiseux germs forms a ring which contains the ring of germs of holomorphic functions. Note that the elementary symmetric polynomials of a Laurent-Puiseux germ and its associates will be single valued and hence meromorphic. Thus the ring of Laurent-Puiseux germs is an algebraic extension of the field of meromorphic germs. Furthermore this may be used to show that the graph of a Laurent-Puiseux germ is an analytic variety in $\Delta_r^* \times \mathbb{C}$, which may be extended to an analytic variety in $\Delta_r \times \hat{\mathbb{C}}$.

Differentiation of Laurent-Puiseux germs is naturally understood locally and can also be done via its Laurent-Puiseux expansion. We may also speak of the evaluation of a Laurent-Puiseux germ f at 0, which is denoted as $f|_{x=0}$. Given the Laurent-Puiseux expansion of f in (A.4), We say $f|_{x=0} = \infty$ if $c_p \neq 0$ for any $p < 0$. If $c_p = 0$ for all $p < 0$, then we say $f|_{x=0} = c_0$.

A family of Laurent-Puiseux germs $\{h_j\}_{j\in J}$ is termed *associate symmetric* or *a.s.* if all associates of each member are also contained with equal multiplicity. Often we will refer to a family of Laurent-Puiseux germs $\{h_j\}_{j\in J}$ tagged with objects a_j

(integers or, most generally, Laurent-Puiseux germs). Such a family is called a.s. if each member h_j with tag a_j implies the inclusion of the kth associate of h_j tagged with the kth associate of a_j, with equal multiplicity. A *distinct* family of Laurent-Puiseux germs $\{h_j\}_{j\in J}$, with or without tags, is one such that $h_j = h_{j'}$ always implies $j = j'$.

First we begin with the following theorem, which is useful as a general tool in this study.

Theorem A.1. *Let $\{h_j\}_{j\in J}$ be a finite family of Laurent-Puiseux germs about $x = 0$, tagged with Laurent-Puiseux germs g_j. If for all but finitely many $m \geq 1$,*

$$\left.\frac{\sum_j g_j(x)h_j(x)^m}{x^{m-1}}\right|_{x=0} = 0,$$

then for all j' such that $\left.\frac{h_{j'}(x)}{x}\right|_{x=0} = \infty$ it holds that

$$\sum_{j \mid h_j = h_{j'}} g_j \equiv 0.$$

The proof of this theorem will require some linear algebra calculations. Define the $n \times n$ matrix $V_{k,\{m_1,\ldots,m_\ell\}}$, where $n = m_1 + \cdots + m_\ell$, as

(A.5) $\quad V_{k,\{m_1,\ldots,m_\ell\}} =$

$$\left[\begin{array}{cccc|ccc}
y_1^k & ky_1^{k-1} & \cdots & \binom{k}{m_1-1}y_1^{k-m_1+1} & y_2^k & \cdots & \binom{k}{m_\ell-1}y_\ell^{k-m_\ell+1} \\
y_1^{k+1} & (k+1)y_1^k & \cdots & \binom{k+1}{m_1-1}y_1^{k-m_1+2} & y_2^{k+1} & \cdots & \binom{k+1}{m_\ell-1}y_\ell^{k-m_\ell+2} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
y_1^{k+n-1} & \binom{k+n-1}{1}y_1^{k+n-2} & \cdots & \binom{k+n-1}{m_1-1}y_1^{k-m_1+n} & y_2^{k+n-1} & \cdots & \binom{k+n-1}{m_\ell-1}y_\ell^{k-m_\ell+n}
\end{array}\right].$$

Lemma A.2.

$$\det V_{k,\{m_1,\ldots,m_\ell\}} = \left(\prod_{j=1}^{\ell} y_j^{km_j}\right)\left(\prod_{1\leq j<i\leq\ell}(y_i - y_j)^{m_jm_i}\right) \tag{A.6}$$

Proof: The determinant of this matrix can be recursively determined by the relation $\det V_{k,\{m_1,m_2,\ldots,m_\ell\}} = y_1^k \left(\prod_{j=2}^{\ell}(y_j - y_1)_j^m\right) \det V_{k,\{m_1-1,m_2,\ldots,m_\ell\}}$. This relation may be derived by simultaneously subtracting from each row the y_1 multiple of the row preceding, factoring each column appropriately, and then calculating by cofactors using the first column. From this relation the lemma easily follows.

$\square$

Proof (of Theorem): Define $G_j(x) = \sum_{j \mid h_j = h_{j'}} g_j$. Without loss of generality, we may assume $G_j \neq 0$ for all $j \in J$.

Suppose for sake of contradiction, there exists a j' such that $\frac{h_j(x)}{x}|_{x=0} = \infty$. Among such j', consider the increasing (possibly finite) sequence of exponents of the Laurent-Puiseux expansion of $G_j$with non-zero coefficients. Now fix j' to be such that this sequence is lexicographically minimal, treating the presence of a sequence term as lexicographically preceding the absence of a sequence term. (e.g. $(1/2, 2, \ldots)$ precedes $(3/4, 3/2, \ldots)$ and $(1/2, 2, \ldots)$ precedes $(1/2, 2)$ precedes $(1/2)$.) Define this lexicographically minimal sequence as $p_1, p_2, \ldots, p_N$ or $p_1, p_2, \ldots$ (defining $N = \infty$).

Define the Laurent-Puiseux expansions of h_j and g_j as

$$h_j(x) = \sum_p c_{j,p} x^p \tag{A.7}$$

$$g_j(x) = \sum_p b_{j,p} x^p \tag{A.8}$$

Let β represent a finite sequence (of length at most N) of non-negative integers, $(b_1, b_2, \ldots, b_\ell)$. Define $|\beta| = \sum_q \beta_q$, $||\beta|| = \sum_q \beta_q p_q$, and $\beta! = \prod_q \beta_q!$.

Define, for $j \in J$, $i, m \geq 0$, $p \in \mathbb{Q}$,

$$s_{j,i,m,p} = \sum_{\beta \mid \beta_1 = \cdots = \beta_i = 0, |\beta| = m, ||\beta|| = p} \frac{m!}{\beta!} \prod_q c_{j,q}^{\beta_q}. \tag{A.9}$$

Note this is well-defined as there are only finitely many β which meet the given criteria. (In fact in most cases there are no such β.) Next define

$$a_{j,i,m,p} = \sum_{p'} b_{j,p-p'} s_{j,i,m,p'}, \tag{A.10}$$

A basic calculation reveals

(A.11)
$$s_{j,i,m,p} = \sum_{k=0}^{m} \binom{m}{k} c_{j,p_{i+1}}^{k} s_{j,i+1,m-k,p-kp_{i+1}} = \sum_{k=0}^{m} \binom{m}{k} c_{j,p_{i+1}}^{m-k} s_{j,i+1,k,p-(m-k)p_{i+1}}.$$

and

$$a_{j,i,m,p} = \sum_{k=0}^{m} \binom{m}{k} c_{j,p_{i+1}}^{m-k} a_{j,i+1,k,p-(m-k)p_{i+1}}. \tag{A.12}$$

The definitions of $s_{j,i,m,p}$ and $a_{j,i,m,p}$ when $i = 0$ provide the following Laurent-Puiseux expansions.

$$h_j(x)^m = \sum_p s_{j,0,m,p} x^p \tag{A.13}$$

$$g_j(x) h_j(x)^m = \sum_p a_{j,0,m,p} x^p \tag{A.14}$$

The initial assumption of the theorem implies that $\sum_j a_{j,0,m,p} = 0$ for $p \leq m - 1$, except for finitely many values of m. Equation (A.12) then provides

$$\begin{aligned} \sum_{j \in J} a_{j,0,m+\kappa,p+\kappa p_1} &= \sum_{j \in J} \sum_{k=0}^{m+\kappa} \left[\binom{m+\kappa}{k} c_{j,p_1}^{m+\kappa-k} a_{j,1,k,p+(k-m)p_1} \right] \\ &= \sum_{\lambda \in \mathbb{C}} \sum_{k=0}^{m+\kappa} \left[\binom{m+\kappa}{k} \lambda^{m+\kappa-k} \left(\sum_{j \mid c_{j,p_1} = \lambda} a_{j,1,k,(p-mp_1)+kp_1} \right) \right]. \end{aligned} \tag{A.15}$$

Assume m and p are fixed. For κ arbitrarily large, the LHS of the above is always zero (by the theorem's initial assumption and since $p_1 < 1$). By the minimality of p_1, it holds that $a_{j,1,k,(p-mp_1)+kp_1}$ is zero for all j for sufficiently large k. Thus we have an infinite set of homogeneous linear equations on terms of the form

$\sum_{j \mid c_{j,p_1}=\lambda} a_{j,1,k,(p-mp_1)+kp_1}$. A well selected subset of these equations will imply that a column vector containing entries $\sum_{j \mid c_{j,p_1}=\lambda} a_{j,1,k,p+(k-m)p_1}$, for the finitely many relevant values of k and λ, is in the null space of a matrix of the form (A.5), the determinant of which is calculated by Lemma A.2. It follows that

$$\sum_{j \mid c_{j,p_1}=\lambda} a_{j,1,m,p} = 0 \tag{A.16}$$

for all m, p, and $\lambda \neq 0$.

We proceed with an induction argument, using the above as the base case. Assume that it has been established that

$$\sum_{j \mid \forall i' \leq i, c_{j,p_{i'}}=\lambda_{i'}} a_{j,i,m,p} = 0, \tag{A.17}$$

for all m, p, and nonzero $\lambda_1, \ldots, \lambda_i$. Then

$$\begin{aligned} 0 &= \sum_{j \mid \forall i' \leq i, c_{j,p_{i'}}=\lambda_{i'}} a_{j,i,m+\kappa,p+\kappa p_{i+1}} \\ &= \sum_{j \mid \forall i' \leq i, c_{j,p_{i'}}=\lambda_{i'}} \sum_{k=0}^{m+\kappa} \left[\binom{m+\kappa}{k} c_{j,p_{i+1}}^{m+\kappa-k} a_{j,i+1,k,p+(k-m)p_{i+1}} \right] \\ &= \sum_{\lambda_{i+1} \in \mathbb{C}} \sum_{k=0}^{m+\kappa} \left[\binom{m+\kappa}{k} \lambda_{i+1}^{m+\kappa-k} \left(\sum_{j \mid \forall i' \leq i+1, c_{j,p_{i'}}=\lambda_{i'}} a_{j,i+1,k,p+(k-m)p_{i+1}} \right) \right]. \end{aligned} \tag{A.18}$$

Assume m and p are fixed. The expression $\sum_{j \mid \forall i' \leq i+1, c_{j,p_{i'}}=\lambda_{i'}} a_{j,i+1,k,p+(k-m)p_{i+1}}$ is zero for large enough values of k. By application of linear algebra and Lemma A.2, we derive (A.17) with $i+1$ in the place of i. Therefore we inductively conclude that (A.17) holds for all values of $i \leq N$.

This holds in particular when $m = 0$. Then note $a_{j,i,0,p} = b_{j,p}$. By the choice of $p_1, p_2, \ldots, p_N$ (or $p_1, p_2, \ldots$) we conclude that

$$\sum_{j \mid h_j = h_{j'}} b_{j,p} = \sum_{j \mid \forall i' \leq i, c_{j,p_{i'}} = c_{j',p_{i'}}} b_{j,p} = 0, \tag{A.19}$$

for all p. Thus $G_{j'} \equiv 0$ which leads to a contradiction.

□

A result related to this is the following.

Theorem A.3. *Let $\{h_j\}_{j \in J}$ be a finite family of Laurent-Puiseux germs about $x = 0$, tagged with Laurent-Puiseux germs g_j such that $(x g_j(x))\big|_{x=0} = 0$. The following are equivalent:*

1. *For all j' such that $\left.\frac{h_{j'}(x)}{x}\right|_{x=0} = \infty$ it holds that $\sum_{j \mid h_j = h_{j'}} g_j \equiv 0$.*
2. *For all $m \geq 1$, $\left.\frac{\sum_{j \in J} g_j(x) h_j(x)^m}{x^{m-1}}\right|_{x=0} = 0$.*
3. *For all (m, ℓ) with $m > \ell \geq 0$, $\left.\left(\sum_{j \in J} g_j(x) h_j(x)^m\right)^{(\ell)}\right|_{x=0} = 0$*
4. *For all but finitely many $m \geq 1$, $\left.\frac{\sum_{j \in J} g_j(x) h_j(x)^m}{x^{m-1}}\right|_{x=0} = 0$.*
5. *For all but finitely many (m, ℓ) with $m > \ell \geq 0$, $\left.\left(\sum_{j \in J} g_j(x) h_j(x)^m\right)^{(\ell)}\right|_{x=0} = 0$*

Proof: It clearly holds that $2 \iff 3 \implies 4 \iff 5$. Theorem A.1 gives that $4 \implies 1$. We point out that

$$\frac{\sum_{j \in J} g_j(x) h_j(x)^m}{x^{m-1}} = \sum_{j \in J} (x g_j(x)) \left(\frac{h_j(x)}{x}\right)^m. \tag{A.20}$$

With this one can see that $1 \implies 2$.

□

We make a couple of remarks here. When we are dealing with associate symmetric families, then $\sum_{j \in J} g_j(x) h_j(x)^m$ is a (single-valued) meromorphic function for all m.

When dealing with general Laurent-Puiseux families, conditions 2 and 4 are more natural and simpler to handle than conditions 3 and 5.

However in some applications the sums $\sum_{j\in J} g_j(x)h_j(x)^m$ are known in advance to be holomorphic. In this case verification of 3 and 5 are more computationally direct. For they can be interpreted as evaluation at $x = 0$ in the sense of holomorphic germs. In contrast, conditions 2 and 4 would still require evaluation of $\frac{\sum_{j\in J} g_j(x)h_j(x)^m}{x^{m-1}}$ in the sense of Laurent-Puiseux germs, which necessitates an accompanying holomorphicity test.

The next two theorems are applications of Theorem A.1 and Theorem A.3. But first we wish to ensure the meaning of some terminology. As mentioned previously the graph of a Laurent-Puiseux germ gives a variety in $\Delta_r \times \hat{\mathbb{C}}$, for some appropriately small r. Also we can think of the line $\{x = 0\}$ as its projective version in $\Delta_r \times \hat{\mathbb{C}}$. In this thinking meromorphic germs yield graphs which intersect the line $\{x = 0\}$ at ∞, though they would not otherwise intersect if we were looking only in $\mathbb{C}^2$. We will refer to the projective line $\{x = 0\}$ to distinguish that we mean the line $\{x = 0\}$ in $\Delta_r \times \hat{\mathbb{C}}$.

Also recall the notion of non-tangential contact. In this context, we say that a variety intersects a line with *non-tangential contact* if at the points of intersections, the tangent cone of the variety trivially intersects the tangent plane of the line. (We previously gave the notion of non-tangential contact in Chapter V.)

Theorem A.4. *Let $\{h_j\}_{j\in J}$ be a finite distinct family of Laurent-Puiseux germs about $x = 0$ tagged with integer multiplicities μ_j.*

The following are equivalent:

1. *The graphs of germs tagged with non-zero multiplicities from the family of germs $\{h_j\}_{j\in J}$ intersect the projective line $\{x = 0\}$ at most at 0 with non-tangential*

contact.

2. $\forall j$ *such that* $\left.\frac{h_j(x)}{x}\right|_{x=0} = \infty$, $\mu_j = 0$

3. *For all* $m \geq 1$, $\left.\frac{\sum_{j\in J}\mu_j h_j(x)^m}{x^{m-1}}\right|_{x=0} = 0$.

4. *For all* (m, ℓ) *with* $m > \ell \geq 0$, $\left.(\sum_j \mu_j h_j(x)^m)^{(\ell)}\right|_{x=0} = 0$

5. *For all but finitely many* $m \geq 1$, $\left.\frac{\sum_{j\in J}\mu_j h_j(x)^m}{x^{m-1}}\right|_{x=0} = 0$.

6. *For all but finitely many* (m, ℓ) *with* $m > \ell \geq 0$, $\left.(\sum_j \mu_j h_j(x)^m)^{(\ell)}\right|_{x=0} = 0$.

Proof: Condition 1 is equivalent to $\forall j \in J$, $\mu_j = 0$ or h_j is $O(x)$ as $x \to 0$. This is equivalent to 2. To complete the proof, Theorem A.3 implies that 2, 3, 4, 5, and 6 are all equivalent.

□

Theorem A.5. *Let* $\{h_j\}_{j\in J}$ *be a finite distinct family of Laurent-Puiseux germs about* $x = 0$ *with associated multiplicities* μ_j. *Let* $a_1, a_2, \ldots, a_n$ *be distinct points in* $\mathbb{C}$.

The following are equivalent:

1. *The graphs of germs tagged with non-zero multiplicity from the family of germs* $\{h_j\}_{j\in J}$ *only intersect the projective line* $\{x = 0\}$ *at most at* $a_1, a_2, \ldots, a_n$ *with non-tangential contact.*

2. $\forall j$ *such that* $\left.\frac{h_j(x)-a_k}{x}\right|_{x=0} = \infty$ *for all* k, $\mu_j = 0$

3. $\forall j$ *such that* $\left.\frac{\prod_{k=1}^n (h_j(x)-a_k)}{x}\right|_{x=0} = \infty$, $\mu_j = 0$

4. *For all* $(m_1, m_2, \ldots, m_n)$ *with* $m_k \geq 1$ *for all* k,
   $\left.\frac{\sum_j \mu_j \prod_{k=1}^n (h_j(x)-a_k)^{m_k}}{x^{m-1}}\right|_{x=0} = 0$, *where* $m = \min\{m_1, m_2, \ldots, m_n\}$.

5. For all $(m_1, m_2, \ldots, m_n, \ell)$ *with* $m_k > \ell \geq 0$ *for all* k,

$\left(\sum_j \mu_j \prod_{k=1}^{n} (h_j(x) - a_k)^{m_k}\right)^{(\ell)}\Big|_{x=0} = 0$

6. For all k',m,ℓ *with* $m > \ell \geq 0$ *and* $0 \leq k' \leq n-1$,

$\left(\sum_j \mu_j \prod_{k=1}^{n} (h_j(x) - a_k)^{m_k}\right)^{(\ell)}\Big|_{x=0} = 0,$ *where* $m_k = \begin{cases} m+1 & \text{if } k \leq k' \\ m & \text{if } k > k' \end{cases}$.

7. For all but finitely many $(m_1, m_2, \ldots, m_n)$ *with* $m_k \geq 1$ *for all* k,

$\frac{\sum_j \mu_j \prod_{k=1}^{n} (h_j(x) - a_k)^{m_k}}{x^{m-1}}\Big|_{x=0} = 0,$ *where* $m = \min\{m_1, m_2, \ldots, m_n\}$.

8. For all but finitely many $(m_1, m_2, \ldots, m_n, \ell)$ *with* $m_k > \ell \geq 0$ *for all* k,

$\left(\sum_j \mu_j \prod_{k=1}^{n} (h_j(x) - a_k)^{m_k}\right)^{(\ell)}\Big|_{x=0} = 0$

9. For all but finitely many k',m,ℓ *with* $m > \ell \geq 0$ *and* $0 \leq k' \leq n-1$,

$\left(\sum_j \mu_j \prod_{k=1}^{n} (h_j(x) - a_k)^{m_k}\right)^{(\ell)}\Big|_{x=0} = 0,$ *where* $m_k = \begin{cases} m+1 & \text{if } k \leq k' \\ m & \text{if } k > k' \end{cases}$.

Proof: Condition 1 is equivalent to $\forall j \in J$, $\mu_j = 0$ or $h_j - a_k$ is $O(x)$ as $x \to 0$ for some k. This is equivalent to condition 2. And as the a_k are distinct, this is also equivalent to $\forall j \in J$, $\mu_j = 0$ or $\prod_{k=1}^{n}(h_j(x) - a_k)$ is $O(x)$ as $x \to 0$. This is equivalent to 3.

The following set of implications are clear.

$$\begin{array}{ccccc} (4) & \Longleftrightarrow & (5) & \Longrightarrow & (6) \\ \Downarrow & & \Downarrow & & \Downarrow \\ (7) & \Longleftrightarrow & (8) & \Longrightarrow & (9) \end{array}$$

We remark that $6 \implies 5$ can be shown via direct algebraic manipulation, and similarly so for $9 \implies 8$. However such implications are not needed as this proof concludes by showing $9 \implies 3 \implies 4$.

Consider the family of Laurent-Puiseux germs $\{\prod_{k=1}^{n}(h_j(x)-a_k)\}_{j\in J}$ tagged with Laurent-Puiseux germs $\mu_j\prod_{k=1}^{n}(h_j(x)-a_k)^{\delta_k}$, for some non-negative integers $\delta_1,\ldots\delta_n$. Applying Theorem A.3 (in particular using its implication $1 \implies 2$) with this new family proves $3 \implies 4$ of the present theorem.

Now assume 9 holds and apply Theorem A.1 to this new family with $\delta_k = m_k - m$. This shows that for all j such that $\left.\frac{\prod_{k=1}^{n}(h_j(x)-a_k)}{x}\right|_{x=0} = \infty$, then

$$\sum_{j'\in J'_j}\mu_{j'}\prod_{k=1}^{k'}(h_{j'}(x)-a_k)\equiv 0,\ \text{ for all } k',\ 0\le k'<n, \tag{A.21}$$

where $J'_j = \{j' \mid \prod_{k=1}^{n}(h_{j'}(x)-a_k) = \prod_{k=1}^{n}(h_j(x)-a_k)\}$. (If $k'=0$, then we interpret the product in (A.21) as equaling 1.) By linear combinations of (A.21), we derive that

$$\sum_{j'\in J'_j}\mu_{j'}h_{j'}(x)^{k'}\equiv 0,\ \text{ for all } k',\ 0\le k'<n. \tag{A.22}$$

Note that $|J'_j|\le n$. From the above equations we conclude that $\mu_{j'}=0$ for all $j'\in J_j$, and so for $j'=j$ in particular. Thus condition 3 follows.

$\square$

APPENDIX B

Formulas for $p_{k,\ell}$

In this section we'll always assume that

$$(P_{i+1})_\xi = \mu P_i - (P_i)_\eta, \text{ for } 0 \le i \le N, \qquad (P_0)_\xi = 0, \tag{B.1}$$

where in any context N is chosen large enough so that we assume that all equations hold that pertain to any P_i being used.

Recall $\mathfrak{U}$ is the algebra of formal differential (with respect to ξ and η) expressions of $\mu_\xi = D_\xi \mu$. By Lemma VIII.1 we know there exist $\rho_{k,\ell,i,j} \in \mathfrak{U}$, for $0 \le j \le i < k < \ell$ such that

$$D_\xi^\ell P_k = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \rho_{k,\ell,i,j} D_\xi^j P_i \tag{B.2}$$

for $\ell > k \le 0$. The purpose of this section is demonstrate technical recurrence relations with which $\rho_{k,\ell,i,j}$ can be calculated.

The equations and derivations will be somewhat simpler if we define the $\rho_{k,\ell,i,j}$ on a broader set of (k, ℓ, i, j). We extend these definitions via a sequence of successive, natural-looking extensions. For one we will define $\rho_{k,\ell,i,j}$ to be defined as 0, whenever $j < 0$, $i < j$, or $i \ge k$, that is whenever $0 \le j \le i < k$ doesn't hold. So, for $0 \le k < \ell$, we can broaden (B.2) to the representation

$$D_\xi^\ell P_k = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \rho_{k,\ell,i,j} D_\xi^j P_i, \tag{B.3}$$

or even

$$D_\xi^\ell P_k = \sum_i \sum_j \rho_{k,\ell,i,j} D_\xi^j P_i. \tag{B.4}$$

Secondly note the above equation still is suitable when $0 \leq \ell \leq k$, namely by setting $\rho_{k,\ell,i,j}$ to be 1 if and only if $i = k$ and $j = \ell$ and 0 otherwise. Thirdly we define $\rho_{k,\ell,i,j}$ to be 0 if $k < 0$ or $\ell < 0$. Thus $\rho_{k,\ell,i,j}$ will be defined for all choices of integer four-tuples (k, ℓ, i, j).

Also as a matter of simplicity of notation and for unification of cases, we permit summations of form $\sum_{j=a}^{a-1}$ as valid notation and to be simply interpreted as 0. We also use the Kronecker delta, $\delta_p = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \end{cases}$. (We also discuss these notational items in Chapter VIII following Lemma VIII.1.) We now provide the following definition for $\rho_{k,\ell,i,j}$.

Theorem B.1. *Equation (B.2) is satisfied by $\rho_{k,\ell,i,j}$ which can be well-defined according to the recursive rule*

$$\begin{aligned} \rho_{k,\ell,i,j} = \sum_{j_1=k}^{\ell-2} \binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu)\rho_{k-1,j_1,i,j} - \sum_{j_1=j+1}^{i} \binom{j_1}{j} (D_\xi^{j_1-j}\mu)\rho_{k-1,\ell-1,i,j_1} \\ + \binom{\ell-1}{j} (D_\xi^{\ell-1-j}\mu)\delta_{k-1-i} - (D_\eta \rho_{k-1,\ell-1,i,j}) + \rho_{k-1,\ell-1,i-1,j-1}, \end{aligned} \tag{B.5}$$

for $0 \leq j \leq i < k < \ell$, where we also understand the following base definitions for $\rho_{k,\ell,i,j}$.

- *For $k < 0$ or $\ell < 0$, $\rho_{k,\ell,i,j} = 0$.*
- *For $0 \leq \ell \leq k$, $\rho_{k,\ell,i,j} = \delta_{k-i}\delta_{\ell-j}$.*
- *For $0 \leq k < \ell$ and any of $j < 0$, $i < j$, or $i \geq k$, $\rho_{k,\ell,i,j} = 0$.*

Proof: First we show that the recursive rules constitute a well-defined definition. Noting that $\rho_{k,\ell,i,j}$ will only recursively depend on other ρ of the form $\rho_{k-1,\ell',i',j'}$, and that for $k = 0$ the base definitions must apply, it holds inductively that $\rho_{k,\ell,i,j}$ for $k, \ell \geq 0$ is well-defined.

Now it simply remains to demonstrate that this definition satisfies (B.2). With aid from the identity in (8.3), and by copious manipulation of summations, this is accomplished by the following calculation, for $0 < k < \ell$.

$$
\begin{aligned}
D_\xi^\ell P_k &= D_\xi^{\ell-1}(\mu P_{k-1} - D_\eta P_{k-1}) \\
&= \sum_{j_1=k}^{\ell-2} \left[\binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu)(D_\xi^{j_1} P_{k-1}) \right] \\
&\qquad + \sum_{j_1=0}^{k-1} \left[\binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu)(D_\xi^{j_1} P_{k-1}) \right] + (\mu - D_\eta)(D_\xi^{\ell-1} P_{k-1}) \\
&= \sum_{i=0}^{k-2} \sum_{j=0}^{i} \left[\sum_{j_1=k}^{\ell-2} \binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu) \rho_{k-1,j_1,i,j} (D_\xi^j P_i) \right] \\
&\qquad + \sum_{i=0}^{k-1} \sum_{j=0}^{i} \left[\delta_{k-1-i} \binom{\ell-1}{j} (D_\xi^{\ell-1-j}\mu)(D_\xi^j P_i) \right] \\
&\qquad - \sum_{i=0}^{k-2} \sum_{j=0}^{i} \left[(D_\eta \rho_{k-1,\ell-1,i,j})(D_\xi^j P_i) \right] \\
&\qquad + \sum_{i=0}^{k-2} \sum_{j_1=0}^{i} \left[\rho_{k-1,\ell-1,i,j_1} \left((\mu - D_\eta) D_\xi^{j_1} P_i \right) \right] \\
&= \sum_{i=0}^{k-1} \sum_{j=0}^{i} \Bigg[\Bigg(\sum_{j_1=k}^{\ell-2} \binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu) \rho_{k-1,j_1,i,j} \\
&\qquad + \delta_{k-1-i} \binom{\ell-1}{j} (D_\xi^{\ell-1-j}\mu) - (D_\eta \rho_{k-1,\ell-1,i,j}) \Bigg) D_\xi^j P_i \Bigg] \\
&\qquad + \sum_{i=0}^{k-2} \sum_{j_1=0}^{i} \left[\rho_{k-1,\ell-1,i,j_1} \left(D_\xi^{j_1+1} P_{i+1} - \sum_{j=0}^{j_1-1} \binom{j_1}{j} (D_\xi^{j_1-j}\mu)(D_\xi^j P_i) \right) \right]
\end{aligned}
$$

$$= \sum_{i=0}^{k-1} \sum_{j=0}^{i} \Bigg[\Bigg(\rho_{k,\ell,i,j} + \sum_{j_1=j+1}^{i} \binom{j_1}{j} (D_\xi^{j_1-j}\mu) \rho_{k-1,\ell-1,i,j_1} - \rho_{k-1,\ell-1,i-1,j-1} \Bigg) D_\xi^j P_i \Bigg]$$

$$+ \sum_{i=1}^{k-1} \sum_{j_1=1}^{i} \left[\rho_{k-1,\ell-1,i-1,j_1-1} D_\xi^{j_1} P_i \right]$$

$$- \sum_{i=0}^{k-2} \sum_{j=0}^{i-1} \left[\left(\sum_{j_1=j+1}^{i} \rho_{k-1,\ell-1,i,j_1} \binom{j_1}{j} (D_\xi^{j_1-j}\mu) \right) D_\xi^j P_i \right]$$

$$= \sum_{i=0}^{k-1} \sum_{j=0}^{i} \left[\rho_{k,\ell,i,j} D_\xi^j P_i \right] \tag{B.6}$$

$\square$

The above theorem is essentially a result of pursuing the proof of Lemma VIII.1 in an computational, rather than simply an existential, fashion. However this does not mean that the rules and definitions so derived are the simplest or most efficient. One helpful reduction results from the following identity.

Theorem B.2. *For $0 \leq j \leq i < k < \ell$,*

$$\rho_{k,\ell,i,j} = \binom{\ell}{j} \rho_{k-j,\ell-j,i-j,0}. \tag{B.7}$$

Proof: We proceed by induction on $k - i$. First consider the case $k = i + 1$. This implies that several of terms in (B.5) will vanish and thus

$$\rho_{i+1,\ell,i,j} = \binom{\ell-1}{j} (D_\xi^{\ell-1-j}\mu) + \rho_{i,\ell-1,i-1,j-1}. \tag{B.8}$$

Note that the above equation can be applied to it's rightmost term. So by inductively applying this equation $j + 1$ times we get that

$$\rho_{i+1,\ell,i,j} = \sum_{j_2=0}^{j} \binom{\ell - j_2 - 1}{j - j_2} (D_\xi^{\ell-1-j}\mu) = \binom{\ell}{j} D_\xi^{\ell-1-j}\mu \tag{B.9}$$

Note that this equation also implies that $\rho_{i+1-j,\ell-j,i-j,0} = D_\xi^{\ell-1-j}\mu$. Thus (B.7) holds in the case $k - i = 1$.

Now assume that we've proven (B.7) whenever $k - i < m$ where $m \geq 2$. Now assume that $k - i = m$. Then (B.5) yields

$$\text{(B.10)}\quad \rho_{k,\ell,i,j} = \sum_{j_1=k}^{\ell-2} \binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu)\rho_{k-1,j_1,i,j} - \sum_{j_1=j+1}^{i} \binom{j_1}{j} (D_\xi^{j_1-j}\mu)\rho_{k-1,\ell-1,i,j_1} - (D_\eta \rho_{k-1,\ell-1,i,j}) + \rho_{k-1,\ell-1,i-1,j-1},$$

Since $(k-1)-(i-1) = m$, we can re-apply this equation to the rightmost term. Applying this equation inductively $j+1$ times yields that

$$\text{(B.11)}\quad \rho_{k,\ell,i,j} = \sum_{j_2=0}^{j} \Bigg\{ \sum_{j_1=k-j_2}^{\ell-j_2-2} \left[\binom{\ell-j_2-1}{j_1} (D_\xi^{\ell-j_2-1-j_1}\mu)\rho_{k-j_2-1,j_1,i-j_2,j-j_2} \right] - \sum_{j_1=j-j_2+1}^{i-j_2} \left[\binom{j_1}{j-j_2} (D_\xi^{j_1-j+j_2}\mu)\rho_{k-j_2-1,\ell-j_2-1,i-j_2,j_1} \right] - D_\eta \rho_{k-j_2-1,\ell-j_2-1,i-j_2,j-j_2} \Bigg\}$$

Now our inductive hypothesis applies to all the terms above. Then applying (B.7) to above yields that

$$\rho_{k,\ell,i,j} = \sum_{j_2=0}^{j} \Bigg\{ \sum_{j_1=k}^{\ell-2} \left[\binom{\ell-j_2-1}{j_1-j_2} \binom{j_1-j_2}{j-j_2} (D_\xi^{\ell-1-j_1}\mu)\rho_{k-j-1,j_1-j,i-j,0} \right] - \sum_{j_1=j+1}^{i} \left[\binom{j_1-j_2}{j-j_2} \binom{\ell-j_2-1}{j_1-j_2} (D_\xi^{j_1-j}\mu)\rho_{k-j_1-1,\ell-j_1-1,i-j_1,0} \right] - \binom{\ell-j_2-1}{j-j_2} D_\eta \rho_{k-j-1,\ell-j-1,i-j,0} \Bigg\}$$

$$= \sum_{j_2=0}^{j} \binom{\ell - j_2 - 1}{j - j_2} \Bigg\{ \sum_{j_1=k}^{\ell-2} \left[\binom{\ell - j - 1}{j_1 - j} (D_\xi^{\ell-1-j_1}\mu) \rho_{k-j-1,j_1-j,i-j,0} \right] - \sum_{j_1=j+1}^{i} \left[\binom{\ell - j - 1}{j_1 - j} (D_\xi^{j_1-j}\mu) \rho_{k-j_1-1,\ell-j_1-1,i-j_1,0} \right] - D_\eta \rho_{k-j-1,\ell-j-1,i-j,0} \Bigg\}$$

$$= \binom{\ell}{j} \Bigg\{ \sum_{j_1=k-j}^{\ell-j-2} \left[\binom{\ell - j - 1}{j_1} (D_\xi^{\ell-j-1-j_1}\mu) \rho_{k-j-1,j_1,i-j,0} \right] - \sum_{j_1=1}^{i-j} \left[(D_\xi^{j_1}\mu) \rho_{k-j-1,\ell-j-1,i-j,j_1} \right] - D_\eta \rho_{k-j-1,\ell-j-1,i-j,0} \Bigg\}$$

$$= \binom{\ell}{j} \rho_{k-j,\ell-j,i-j,0} \tag{B.12}$$

□

Define $\psi_{k,\ell,i} = \rho_{k,\ell,i,0}$. So by Theorem B.2, we only need be concerned with defining $\psi_{k,\ell,i}$ for $0 \le i < k < \ell$. Using (B.5) and (B.7) then $\psi_{k,\ell,i}$ could be defined, for $0 \le i < k < \ell$, by the recursive rule

$$\psi_{k,\ell,i} = \sum_{j_1=k}^{\ell-2} \left[\binom{\ell - 1}{j_1} (D_\xi^{\ell-1-j_1}\mu) \psi_{k-1,j_1,i} \right] - \sum_{j_1=1}^{i} \left[\binom{\ell - 1}{j_1} (D_\xi^{j_1}\mu) \psi_{k-1-j_1,\ell-1-j_1,i-j_1} \right] + (D_\xi^{\ell-1}\mu)\delta_{k-1-i} - (D_\eta \psi_{k-1,\ell-1,i}), \tag{B.13}$$

and the following definitions, (the third of which serves as a base for the above recursive rule)

- For $k < 0$ or $\ell < 0$, $\psi_{k,\ell,i} = 0$.
- For $0 \le \ell \le k$, $\psi_{k,\ell,i} = \delta_{k-i}\delta_\ell$.
- For $0 \le k < \ell$ and any of $i < 0$ or $i \ge k$, $\psi_{k,\ell,i} = 0$.

In Chapter VIII we only have need to know $\rho_{k,k+1,i,j}$ for $0 \leq j \leq i < k$ and thus $\psi_{k,k+1,i}$ for $0 \leq i < k$. The recursive rule restricted to this case is simply

$$\text{(B.14)} \quad \psi_{k,k+1,i} = -\sum_{j_1=1}^{i} \left[\binom{k}{j_1} (D_\xi^{j_1}\mu) \psi_{k-1-j_1,k-j_1,i-j_1} \right] - (D_\eta \psi_{k-1,k,i}) + (D_\xi^k \mu)\delta_{k-1-i}.$$

This gives a suitable recursive definition when coupled with the base definition $\psi_{k,k+1,i} = 0$ for $i \geq k$.

An alternate recursive definition is this case is to define $\psi_{k,k+1,k-1} = D_\xi^k \mu$, for $k \geq 1$, and to recursively define

$$\text{(B.15)} \qquad \psi_{k,k+1,i} = -\sum_{j_1=1}^{i} \left[\binom{k}{j_1} (D_\xi^{j_1}\mu) \psi_{k-1-j_1,k-j_1,i-j_1} \right] - (D_\eta \psi_{k-1,k,i}),$$

for $0 \leq i < k-1$.

APPENDIX C

Linear Dependence of Analytic Functions in Several Variables

The study of linear dependence is a point of general interest. We will look at the specific case of analytic functions in several variables. The case of analytic functions in two variables holds some application in this thesis.

Let Ω be a domain in $\mathbb{C}^m$, for which we use coordinates $x_1, x_2, \ldots, x_m$. Let $f_1, f_2, \ldots, f_n$ be functions defined on Ω. The functions $f_1, f_2, \ldots, f_n$ are linearly dependent over $\mathbb{C}$ if and only exists constants $c_1, c_2, \ldots, c_n$, not all zero, in $\mathbb{C}$ such that $\sum_j^m c_j f_j = 0$. It is a reasonable question to ask how one determines whether such a set of analytic functions is linear dependent or not.

When $m = 1$, a concise answer is that linear dependence is equivalent to the Wronskian identically vanishing. One treatment is in the century old survey of Bôcher [3] (particularly Section 4). An identically vanishing Wronskian is only a necessary condition when analyticity is not assumed. Results in the non-analytic case typically involve adding conditions to yield sufficiency. Work in this case was initiated by Peano and continued by Bôcher [4], Curtiss [6], Chaundy [5], and Wolsson [29]. Henceforth we restrict our focus to the analytic case. So we suppose our functions $f_1, f_2, \ldots, f_n$ to be analytic.

For general m, a condition equivalent to linear dependence is the vanishing of all generalized Wronskians (which we will later define). Roth presents this criteria for polynomials in [23]. Generalized Wronskians were earlier used by Siegel [25]. An earlier statement on linear dependence with multiple variables was given, without proof, by Kellogg in 1912 [19]. (For a view concerning the non-analytic case see Wolsson [30].)

We present here a new criteria, though independently created for its application in this work, which may be expressed as a sharper version of the condition stemming from Roth's. In particular the identical vanishing of a sharply chosen subset of generalized Wronskians provides a sufficient condition for linear dependence. This specifically chosen generalized Wronskians correlate to generalizations of Young diagrams for general dimension. We will state both the generalized Wronskian criteria and our sharpened result.

We use the following multi-index notation. Define $\mathcal{T}_m$ to be the set of m-tuples of non-negative integers. Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \in \mathcal{T}_m$. Let $|\alpha| = \sum_j^m \alpha_j$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$, and $D^\alpha = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \cdots D_{x_m}^{\alpha_m}$. Let $\phi = [f_1, f_2, \ldots, f_n]$. Let $A = \{a_1, a_2, \ldots, a_k\} \subset (\mathcal{T}_m)^k$. Define

$$M_\phi[A] = \begin{bmatrix} D^{a_1} f_1 & D^{a_1} f_2 & \cdots & D^{a_1} f_n \\ D^{a_2} f_1 & D^{a_2} f_2 & \cdots & D^{a_2} f_n \\ \vdots & \vdots & & \vdots \\ D^{a_k} f_1 & D^{a_k} f_2 & \cdots & D^{a_k} f_n \end{bmatrix} = \begin{bmatrix} D^{a_1}\phi \\ D^{a_2}\phi \\ \vdots \\ D^{a_k}\phi \end{bmatrix}. \tag{C.1}$$

For A of order n, define $W_\phi[A] = \det(M_\phi[A])$. If A can be expressed as $\{a_1, a_2, \ldots, a_n\}$ such that $\forall j \ |a_j| \le j-1$, then we call $W_\phi[A]$ a generalized Wronskian (of the function set $\{f_1, f_2, \ldots, f_n\}$). We state the generalized Wronskian method for determining linear dependence.

Theorem C.1. *The function set $\{f_1, f_2, \ldots, f_n\}$ is linear dependent over $\mathbb{C}$ if and* *only if all of its generalized Wronskians vanish.*

Proof: If the function set $\{f_1, f_2, \ldots, f_n\}$ is linear dependent over $\mathbb{C}$, then there exist complex numbers $c_1, c_2, \ldots, c_n$, not all zero, such that $\sum_j c_j f_j = 0$. By differentiation, it holds that for any $a \in \mathcal{T}_m$, that $\sum_j c_j D^a f_j = 0$. Thus $[c_1 \quad c_2 \quad \cdots \quad c_n]^{\mathrm{T}}$ is a non-trivial null vector to $M_\phi(A)$ for all A, thus all the generalized Wronskians must vanish.

For the converse, see [24] pages 80-83, where the sufficiency of the vanishing of the generalized Wronskians for the linear dependence over $\mathbb{R}$ of rational functions with real coefficients is given. That proof, modified to linear dependence over $\mathbb{C}$ of meromorphic functions, will give the converse.

$\square$

We lexicographically order $\mathcal{T} = \mathcal{T}_m$, denoting $\alpha \preceq \beta$ if α lexicographically precedes or equals β. We also define a partial ordering $\leq$ on $\mathcal{T}$, by the rule $(\alpha_1, \alpha_2, \ldots, \alpha_m) \leq (\beta_1, \beta_2, \ldots, \beta_m)$ if and only if $\alpha_j \leq \beta_j$ for all j. Note $\alpha \leq \beta$ implies $\alpha \preceq \beta$.

We call a set $A \subset \mathcal{T}$ *Youngish* if $a \in A$ and $b \leq a$ implies $b \in A$. For $m = 2$, finite Youngish sets correspond to Young diagrams. If A is Youngish then $W_\phi[A]$ is a generalized Wronskian. Now we state the following sharper version of Theorem C.1.

Theorem C.2. *The function set $\{f_1, f_2, \ldots, f_n\}$ is linear dependent over $\mathbb{C}$ if and only if the generalized Wronskians $W_\phi[A]$ vanish for all Youngish sets $A \subset \mathcal{T}_m$ of order n.*

To facilitate the proof we establish some definitions and lemmas. Let $\mathcal{O} = \mathcal{O}_m$ denote the ring of germs of analytic functions in variables $x_1, x_2, \ldots, x_m$ about the origin. We define the field $\mathcal{M} = \mathsf{ff}(\mathcal{O})$, the fraction field of $\mathcal{O}$. Let M be a $1 \times n$

matrix with entries in $\mathcal{O}$. We may consider M as a $\mathcal{M}$-linear map from $\mathcal{M}^n$ to $\mathcal{M}$. For $\alpha \in \mathcal{T}$, define $M_\alpha = D^\alpha(M)$, where we treat differentiation entry-wise on M.

Let A be a subset of $\mathcal{T}$. Define $\mathcal{N}_A = \bigcap_{\alpha \in A} \ker M_\alpha \subseteq \mathcal{M}^n$.

Lemma C.3. *For a given M there exists a Youngish set $Y \subset \mathcal{T}$ of order at most n such that $\mathcal{N}_Y = \mathcal{N}_\mathcal{T}$.*

Proof: Define $S_\alpha = \{\beta \in \mathcal{T} \mid \beta \prec \alpha\}$. Let $Y = \{\alpha \in \mathcal{T} \mid \mathcal{N}_{S_\alpha} \neq \mathcal{N}_{S_\alpha \cup \{\alpha\}}\} = \{\alpha \in \mathcal{T} \mid \mathcal{N}_{S_\alpha} \nsubseteq \ker M_\alpha\}$. We claim this set is Youngish. Assume not, then there exists a $\gamma \in Y$ and $\beta \notin Y$ such that $\beta < \gamma$. Let δ denote the difference between γ and β (a.k.a. $\gamma - \beta$). Now $\beta \notin Y$ implies $\mathcal{N}_{S_\beta} \subseteq \ker M_\beta$. This implies that there exists $b \neq 0$ and a_α, for $\alpha \in S_\beta$, all in $\mathcal{O}$ such that $bM_\beta = \sum_{\alpha \in S_\beta} a_\alpha M_\alpha$. Applying D^δ to this equation yields that bM_γ is an $\mathcal{O}$-linear combination of M_α, $\alpha \in S_\gamma$. This implies $\mathcal{N}_{S_\gamma} \subseteq \ker M_\gamma$, which contradicts γ being in Y. Thus Y is Youngish.

Now $\mathcal{N}_\mathcal{T} \subseteq \mathcal{N}_Y$ is clear. So for contradiction assume $\mathcal{N}_Y \nsubseteq \mathcal{N}_\mathcal{T}$. As $\mathcal{T}$ is well-ordered, let β be the least element in $\mathcal{T}$ such that $\mathcal{N}_Y \nsubseteq \mathcal{N}_{S_\beta \cup \{\beta\}}$. So $\mathcal{N}_Y \subseteq \bigcap_{\alpha \prec \beta} \mathcal{N}_{S_\alpha \cup \{\alpha\}} = \mathcal{N}_{S_\beta}$. If $\beta \notin Y$ then $\mathcal{N}_Y \subseteq \mathcal{N}_{S_\beta} = \mathcal{N}_{S_\beta \cup \{\beta\}}$. If $\beta \in Y$, then $\mathcal{N}_Y \subseteq \mathcal{N}_{S_\beta} \cap \ker M_\beta = \mathcal{N}_{S_\beta \cup \{\beta\}}$. Either way forms a contradiction, thus $\mathcal{N}_Y = \mathcal{N}_\mathcal{T}$.

Now suppose for sake of contradiction that Y contains $n+1$ or more elements. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ be elements of Y such that $\alpha_1 \prec \alpha_2 \prec \ldots \prec \alpha_{n+1}$. Then note

$$\mathcal{M}^n \supset \bigcap_{\alpha \preceq \alpha_1} \ker M_\alpha \supset \bigcap_{\alpha \preceq \alpha_2} \ker M_\alpha \supset \ldots \supset \bigcap_{\alpha \preceq \alpha_{n+1}} \ker M_\alpha \tag{C.2}$$

is a strictly decreasing sequence of vector spaces, the last of which has codimension at least $n+1$ within $\mathcal{M}^n$, which yields the desired contradiction.

□

Proof (of Theorem C.2): The identical vanishing of the generalized Wronskians associated with Youngish sets is clearly necessary, so we only need to establish sufficiency.

Let $M = \phi$ and let Y be a Youngish set of order at most n such that $\mathcal{N}_{\mathcal{T}} = \mathcal{N}_Y$, which exists by Lemma C.3. A generalized Wronskian $W_\phi[A]$ identically vanishes if and only if $\mathcal{N}_A \neq \{0\}$. So if Y is of order n then it holds by assumption that $W_\phi[Y] = 0$ and thus $\mathcal{N}_{\mathcal{T}}$ is non-trivial. If Y has order less than n, then it automatically holds that $\mathcal{N}_{\mathcal{T}}$ is non-trivial. Independent of the previous two cases, it follows that $\mathcal{N}_A \neq \{0\}$ for all subsets A of $\mathcal{T}$. In particular, this implies all the generalized Wronskians vanish and so by Theorem C.1, linear dependence follows.

□

This version of determining linear dependence is sharper than the previous result as we only need to check a subset of the generalized Wronskians. This choice of subset is in fact sharp, as we will see by the following.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite Youngish set. For our function set, let $\phi = \{x^{a_1}, x^{a_2}, \ldots, x^{a_n}\}$. As A is Youngish, for any $b = (\beta_1, \beta_2, \ldots, \beta_m) \notin A$ and $a = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in A$ there exists a j such that $\alpha_j < \beta_j$, so $D^b(x^a) \equiv 0$. Thus $D^b\phi \equiv [0\ 0\ \ldots\ 0]$. Thus for any $B \subset N^m$ of order n, other than A, $W_\phi[B]$ must identically vanish. (In contrast, $W_\phi[A]$ will equal a nonzero constant.) The function set ϕ is linearly independent over $\mathbb{C}$. So any subset of generalized Wronskians whose vanishing is sufficient to determine linear dependence must include $W_\phi[A]$, where A is any Youngish set of order n.

Now if we examine this result in terms of computability, the construction of the Youngish set in Lemma C.3 provides an algorithmic way to check for linear dependence. As demonstrated, linear dependence is tantamount to the set $\mathcal{N}_{\mathcal{T}}$ being non-trivial. In fact we can compute the Y constructed in Lemma C.3 and $\mathcal{N}_{\mathcal{T}} = \mathcal{N}_Y$, by the following algorithm. New notation: let $\mathcal{R}_A = \text{span}_{\mathcal{M}}\{M_\alpha\}_{\alpha \in A}$. So $\mathcal{N}_A = \mathcal{R}_A^\perp$.

1. Initialize α to $(0, 0, \ldots, 0)$, A to $\emptyset$, and $\mathcal{R}_A$ to $\{0\}$.

2. If $M_\alpha \notin \mathcal{R}_A$, then set A to $A \cup \{\alpha\}$ and $\mathcal{R}_A$ to $\mathcal{R}_A \oplus_{\mathcal{M}} M_\alpha$.

3. Set β to be the least element (according to the total ordering $\prec$) greater than α, such that $A \cup \{\beta\}$ is "Youngish". If no such β exists skip to step 5.

4. Set α to β and repeat starting at step 2.

5. Set Y to A and $\mathcal{N}_Y$ to $\mathcal{R}_A^\perp$. Calculation complete.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Alexander H., Wermer J., Several Complex Variables and Banach Algebras, 3rd edition, Springer-Verlag, New York, (1998)

[2] Ahlfors L., Complex Analysis, 3rd edition, McGraw-Hill, New York, (1979)

[3] Bôcher M., The Theory of Linear Dependence, Ann. of Math., 2 (1900), 81-96

[4] Bôcher M., Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence, Trans. Amer. Math. Soc., 2 (1901) 139-149

[5] Chaundy T. W., The vanishing of the Wronskian, J. London Math. Soc., 8 (1933) 4-9

[6] Curtiss D. R., The vanishing of the Wronskian and the problem of linear dependence, Math. Annalen, 65 (1908), 282-298

[7] Darboux, Théorie des surfaces, I, 2nd edition, Gauthier-Villars, Paris, (1914)

[8] Dolbeault P., Henkin G., Surfaces de Riemann de bord donne dans $\mathbb{CP}^n$, Contributions to complex analysis and analytic geometry, Aspects of Math., Vieweg, 26 (1994) 163-187

[9] Dolbeault P., Henkin G., Chaînes holomorphes de bord donné dans $\mathbb{CP}^n$, Bull. Soc. Math. France 125 (1997) 383-445

[10] El Kasimi A., Surfaces de Riemann de bord donné dans la Grassmannienne, Bull. Soc. Sci. Lettres Lódź 49, Rech. Deform. 28 (1999) 55-65

[11] El Kasimi A., Problème du bord dans le compactifiés de $\mathbb{C}^n$, C. R. Acad. Sci. Paris 332 (2001) 121-124

[12] Garabedian P., Partial Differential Equations, John Wiley and Sons, New York, (1964)

[13] Griffiths P., Harris J., Principles of Algebraic Geometry, John Wiley and Sons, New York, (1978)

[14] Harris J., Algebraic Geometry, Springer-Verlag, New York, (1992)

[15] Harvey R., Lawson B., On boundaries of complex analytic varieties, I, Ann. of Math. 102 (1975) 233-290

[16] Harvey R., Lawson B., On boundaries of complex analytic varieties, II, Ann. of Math. 106 (1977) 213-238

[17] Harvey R., Lawson B., Complex analytic geometry and measure theory, Proceedings of Symposia in Pure Mathematics, 44 (1986) 261-274

[18] Harvey R., Shiffman B., A characterization of holomorphic chains, Ann. of Math. 99 (1974) 553-587

[19] Kellogg, Comptes rendus des séances de la Soc. Math. de France, 41 (1912), 19-21

[20] Macdonald I. G., Symmetric Functions and Orthogonal Polynomials, University Lecture Series Vol. 12, American Mathematical Society, Providence, (1998)

[21] Muskhelishvili N. I., Singular Integral Equations: Boundary problems of function theory and their application to mathematical physics, P. Noordhoff, Groningen, (1953)

[22] Peano G., Sur le determinant Wronskien, Mathesis 9 (1889) 75-76

[23] Roth K. F., Rational approximations to algebraic numbers, Mathematika 2 (1955) 1-20

[24] Schmidt W., Diophantine Approximation, Lecture Notes in Mathematics, No.785, Springer, (1980)

[25] Siegel C., Über Näherungswerte algbraischer Zahlen, Math. Annalen, 84 (1921), 80-99

[26] Shafarevich, I. R., Basic Algebraic Geometry, Vol. 1, 2nd edition, Springer-Verlag, New York, (1994)

[27] Shiffman B., On the removal of singularities of analytic sets, Mich. Math. J. 15 (1968) 111-120

[28] Wermer J., The hull of a curve in $\mathbb{C}^n$, Ann. of Math. 68 (1958) 550-561

[29] Wolsson K., A condition equivalent to linear dependence for functions with vanishing Wronskian, Linear Algebra Appl. 116 (1989) 1-8

[30] Wolsson K., Linear dependence of a function set of m variables with vanishing generalized Wronskians, Linear Algebra Appl. 117 (1989) 73-80

ABSTRACT

Concerning Characterizations of Boundaries of Holomorphic 1-Chains within Complex Surfaces

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Let X be a complex manifold or an analytic variety and let γ be a closed, oriented, $\mathcal{C}^2$ real 1-chain in X. We say γ is the boundary of an holomorphic 1-chain within X if there is a holomorphic 1-chain V in $X \setminus \operatorname{spt} \gamma$ such that $b[V] = [\gamma]$, in the sense of currents, and $\operatorname{spt} V \Subset X$. We produce a family of new characterizations for boundaries of holomorphic 1-chains within $\mathbb{CP}^2$, related to a previous characterization involving shockwaves. We show that some of these characterizations are computationally accessible and may be tractably tested. By employing birational maps, we further characterize the boundaries of holomorphic 1-chains within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ and $\mathbb{C} \times \hat{\mathbb{C}}$. By a separate approach, we produce a distinctly new vein of characterizations for boundaries of holomorphic 1-chains within $\mathbb{CP}^2$ and $\mathbb{C} \times \hat{\mathbb{C}}$.