RATIONALITY CRITERIA IN DIFFERENTIAL FIELDS OF SEVERAL VARIABLES

RONALD A. WALKER

ABSTRACT. We develop two classes of criteria for multivariable rationality with bounds on degree or other restrictions of the permitted exponent set. The first class is expressed in terms of differential algebra and shows that rational functions with prescribed exponent sets form a differential algebraic variety expressed by a finite collection of explicit differential relations. The second class concerns characterizations in terms of Taylor series coefficients, including one approach using rationality along radial one-dimensional slices, thus it extends a classical one variable result. In the first class, we develop results pertinent to the cases of zero and prime characteristic, while our Taylor coefficient results focus solely on characteristic zero.

1. INTRODUCTION

Criteria for rationality with respect to a single variable have been studied in various ways and contexts. For instance, there are classical results that characterize the Taylor coefficients of rational functions in the case of analytic functions of a single variable [3][pp.321-323]. More recently it has been noted that there exists a differential condition for characterizing when multivariable meromorphic functions are rational with respect to one specified variable [13]. It is natural to inquire about the generalized conditions for rationality with respect to several variables

Already, this suggests two classes of criteria to consider: one class expressible in terms of differential algebra and the other class expressible in algebraic conditions on Taylor series coefficients. In fact, the latter can also be understood within a differential algebraic framework by our introduction of evaluations, which provide a way to generalize Taylor coefficients and the Cauchy problem. This has connections to the notion of local differential algebra studied by Robinson [9].

In regards to differential algebra criteria, one foundational observation is that the question of rationality can be readily expressed as a question of linear dependence. Thus one may employ generalized Wronskians, using the results of [10], to generate a finite set of explicit differential conditions that characterize rational functions

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with bounds on degree or other restrictions on the permitted exponent set. Such is captured by Theorem 3.2 and its ensuing corollaries and remarks, which we express for the case of general characteristic.

For criteria in terms of Taylor coefficients, which we develop in characteristic zero, our main results are Theorem 4.1 and Theorem 4.4. The latter shows in a certain case analogous to analyticity that rationality with a bound on degree along one dimensional radial slices implies multivariable rationality with the same bound in degree. Also, as an aside, we note that some of the criteria for the classical single variable case produce a set of relations that form an infinite Groebner basis.

One application of (and motivation for) rationality criteria occurs in the area of complex analytic geometry, where the rationality of certain multivariate generating functions of moments plays a significant role in the characterization of boundaries of holomorphic chains. In particular, the "closedness" of such criteria fuels various continuation techniques. For an example, the single variable results have been employed in places such as [5] and [13]. Plus a differential characterization of rationality is an analog to the results of [12], wherein the shockwave decomposability of a related generating function of moments (which arises in [4]) is shown to be equivalent to a certain collection of differential relations.

Here is the structure of this article. In Section 2, we introduce our differential algebraic set-up, including an introduction of the notion of evaluations, plus we cover some relevant linear algebra preliminaries. In Section 3 we present differential multivariable criteria for rationality, discussing results pertinent to both the cases of zero characteristic and prime characteristic. In Section 4 we express rationality criteria of Taylor series, with focus simply on the case of characteristic zero.

2. Preliminaries

2.1. **Differential algebra definitions.** Let $\mathcal{M}$ denote a differential field equipped with m pair-wise commuting derivations $D_1, D_2, \ldots, D_m$. Let $\mathcal{C}$ denote the field of constants $\{f \in \mathcal{M} \mid D_j f = 0 \text{ for } j = 1, 2, \ldots, m\}$. (For a thorough introduction to differential algebra see [6].)

Let $\mathbb{N}$ denote the collection of non-negative integers. Per standard multi-index notation, let $D^\alpha = D_1^{a_1} D_2^{a_2} \cdots D_m^{a_m}$, $\xi^\alpha = \xi_1^{a_1} \xi_2^{a_2} \cdots \xi_m^{a_m}$, $|\alpha| = \sum_{j=1}^m a_j$, and $\alpha! = \prod_{j=1}^m a_j!$. Also let $\binom{\alpha}{\beta} = \prod_{j=1}^m \binom{a_j}{b_j}$, with the understanding that the binomial coefficient $\binom{a}{b}$ is interpreted as zero when $b < 0$ or $b > a$.

For $\alpha = (a_1, a_2, \ldots, a_m)$ and $\beta = (b_1, b_2, \ldots, b_m)$ in $\mathbb{N}^m$, we say that $\alpha \leq \beta$ if and only $a_j \leq b_j$ for $j = 1, , 2 \ldots, m$. We will use $\prec$ to denote graded lexicographical order on $\mathbb{N}^m$. In particular $\alpha \prec \beta$ if and only if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and there exists a k such that $a_k < b_k$ and $a_j = b_j$ for $j < k$. This is an example of a monomial ordering, namely it is a well-ordered, total ordering of $\mathbb{N}^m$ such that $\alpha < \beta$ implies $\alpha + \gamma < \beta + \gamma$ for $\gamma \in \mathbb{N}^m$. [2]

For $A \subseteq \mathbb{N}^m$, let $A + \mathbb{N}^m = \{\alpha + \gamma \mid \gamma \in \mathbb{N}^m\} = \{\beta \in \mathbb{N}^m \mid \beta \geq \alpha \text{ for some } \alpha \in A\}$. A set $Y \subseteq \mathbb{N}^m$ is called *Young-like* if $\alpha \in Y$, $\beta \in \mathbb{N}^m$, and $\beta \leq \alpha$ always imply that $\beta \in Y$. [10] In other words Y is Young-like if and only if $(\mathbb{N}^m \backslash Y) + \mathbb{N}^m = \mathbb{N}^m \backslash Y$.

We define an *evaluation on* $\mathcal{M}$ to be a ring homomorphism E from some differential subring Dom_E of $\mathcal{M}$ to $\mathcal{C}$ such that

(1) the fraction field of Dom_E is $\mathcal{M}$,
(2) Dom_E contains $\mathcal{C}$, and
(3) $E(c) = c$ for $c \in \mathcal{C}$.

By setting $E(g/h) = E(g)/E(h)$, one can naturally extend the definition of E to the localization $(\mathrm{Dom}_E \backslash \ker E)^{-1}\mathrm{Dom}_E = \{\frac{g}{h} \mid g, h \in \mathrm{Dom}_E, E(h) \neq 0\}$ in $\mathcal{M}$. So we can assume that Dom_E corresponds to this localization.

The following examples help motivate and illustrate these definitions.

Example 1. Let $\mathcal{M}$ be the field of germs of meromorphic functions about a point $p = (p_1, p_2, \dots, p_n)$ in $\mathbb{C}^n$ with derivations $D_1, D_2, \dots, D_n$ defined by $D_j = \frac{\partial}{\partial z_j}$. The subfield of constants $\mathcal{C}$ corresponds to $\mathbb{C}$ and the mapping $f \mapsto f(p)$ provides a natural evaluation. The domain of this evaluation is the subring of germs of holomorphic functions about p.

Example 2. Consider $\mathcal{M}$ as defined in the previous example, but with only the derivations $D_1, D_2, \dots, D_m$ for $m < n$. Then the subfield of constants $\mathcal{C}$ corresponds to the field of germs of meromorphic functions in variables $(z_{m+1}, \dots, z_n)$ about $(p_{m+1}, p_{m+2}, \dots, p_n)$ in $\mathbb{C}^{n-m}$. We may define one evaluation E by setting $E(f)(z_{m+1}, \dots, z_n) = f(p_1, \dots, p_m, z_{m+1}, \dots, z_n)$, but other evaluations exist. For instance, one could also obtain an evaluation on $\mathcal{M}$ by setting

$$E(f)(z_{m+1}, \dots, z_n) = f(g_1(z_{m+1}, \dots, z_n), \dots, g_m(z_{m+1}, \dots, z_n), z_{m+1}, \dots, z_n),$$

where $(g_1, \dots, g_m)$ is a germ of a holomorphic map between neighborhoods of $\mathbb{C}^{n-m}$ and $\mathbb{C}^m$ that sends $(p_{m+1}, \dots, p_n)$ to $(p_1, \dots, p_m)$.

From a PDE point-of-view, this evaluation corresponds to evaluating f along a non-characteristic surface of dimension $n - m$ and flowing the values along the characteristics of $D_1, D_2, \dots, D_m$. Other examples can be readily constructed in this spirit. Thus an evaluation is not purely a restriction, but rather a "restriction and flow". This is tied to the fact that the field of constants $\mathcal{C}$ is a subfield of $\mathcal{M}$.

Example 3. A derivation and evaluation can be used to algebraically generalize the Cauchy initial value problem. For instance, let $\mathcal{M}$ be a differential field with a derivation D and an evaluation E. (So $(\mathcal{M}, \mathcal{C}, E)$ is a reflexive localized differential field in the terminology of Robinson [9].) Considering only the additive group structure we have the short exact sequence

$$0 \to \mathcal{C} \to \mathrm{Dom}_E \stackrel{D}{\to} D(\mathrm{Dom}_E) \to 0. \tag{1}$$

Because of the existence of E, this splits to produce the reverse short exact sequence

$$0 \leftarrow \mathcal{C} \stackrel{E}{\leftarrow} \mathrm{Dom}_E \stackrel{I}{\leftarrow} D(\mathrm{Dom}_E) \leftarrow 0, \tag{2}$$

where I is defined by $I(D(f)) = f - E(f)$. One can readily verify that I is well-defined. The map I conceptually corresponds to integration. For instance, if $\mathcal{M}$ is the field of germs of meromorphic functions about 0 in $\mathbb{C}$ with $D = \frac{\partial}{\partial z}$ and $E(f) = f(0)$, then $I(f)(z) = \int_0^z f(t)\, dt$.

Because of this splitting, we have that $\mathrm{Dom}_E \cong \mathcal{C} \oplus D(\mathrm{Dom}_E)$. In this setting, the Cauchy problem

$$Df = h, \qquad Ef = g, \tag{3}$$

for $h \in D(\mathrm{Dom}_E)$ and $g \in \mathcal{C}$, has the unique solution $f = Ih + g$. The direct product splitting of Dom_E conveys the fact that each function f in Dom_E can be represented uniquely based on its derivative data $D(f)$ and Cauchy data $E(f)$. This captures the essence of a well-posed Cauchy problem with the exception that continuous dependence on data would require the consideration of topology in order to be adequately expressed.

Example 4. Given any field $\mathcal{F}$ and indeterminates $x_1, x_2, \ldots, x_n$, the field of quotients of formal power series $\mathcal{F}((x_1, x_2, \ldots, x_n))$ with derivations $D_1, D_2, \ldots D_m$ (given by formal differentiation) forms a differential field. If $\operatorname{char} \mathcal{F} = 0$ then one natural evaluation is $E(\sum_{\alpha \geq 0} c_\alpha x^\alpha) = c_0$. However if $\operatorname{char} \mathcal{F} \neq 0$ this fails to be an evaluation.

2.2. **Linear algebra preliminaries.** For sets A and B in $\mathbb{N}^m$, we define an $A \times B$ matrix to be a matrix whose rows and columns are indexed by the elements of A and B, respectively, with row and column indices ordered using $\prec$, unless otherwise specified. For $\alpha \in A$ and $\beta \in B$, let M_α^β denote the entry of M row-referenced by α and column-referenced by β. In the case of row matrices or column matrices, we may omit the row index or column index, respectively. The application of derivations and evaluations to a matrix is understood to be performed entry-wise.

Let ϕ be a $1 \times B$ row matrix, with B finite. Unless otherwise denoted, we work over $\mathcal{M}$. So, for instance, $\ker \phi$ denotes the null space of ϕ in $\mathcal{M}^{|B|}$. For $A \subseteq \mathbb{N}^m$, we define L_A^ϕ entry-wise by saying $(L_A^\phi)_\alpha^\beta = D^\alpha(\phi^\beta)$, with rows ordered by $\prec$. For instance, if $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ with $\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_n$, then

$$L_A^\phi = \begin{bmatrix} D^{\alpha_1}\phi \\ D^{\alpha_2}\phi \\ \vdots \\ D^{\alpha_n}\phi \end{bmatrix}. \tag{4}$$

When $|A| = |B|$, we define the generalized Wronskian of ϕ associated with the set A to be

$$W_A^\phi = \det L_A^\phi. \tag{5}$$

On some occasions we may wish to modify the order of the rows of L_A^ϕ. To aid this purpose, let P_{C_1,C_2} (with $C_1 \cap C_2 = \emptyset$) be the $(C_1 \cup C_2) \times (C_1 \cup C_2)$ permutation matrix such that left-multiplication rearranges rows so that those indexed by C_1 are placed first (ordered among themselves by $\prec$) followed by the rows indexed by C_2 (also ordered by $\prec$).

The linear dependence results of [10] have arguments that readily carry over to the abstract differential algebraic framework considered here. We restate two such results that we will employ later.

Lemma 2.1. *Let ϕ be a $1 \times B$ matrix with entries in $\mathcal{M}$, where B is some finite index set. It holds that any reduced $\mathcal{M}$-basis of $\bigcap_{\alpha \in \mathbb{N}^m} \ker D^\alpha \phi$ lies within $\mathcal{C}^{|B|}$, and thus*

$$\bigcap_{\alpha \in \mathbb{N}^m} \ker D^\alpha \phi = \left(\bigcap_{\alpha \in \mathbb{N}^m} \ker D^\alpha \phi \cap \mathcal{C}^{|B|} \right) \otimes_{\mathcal{C}} \mathcal{M} = \left(\ker \phi \cap \mathcal{C}^{|B|} \right) \otimes_{\mathcal{C}} \mathcal{M} \tag{6}$$

Theorem 2.2. *Let ϕ be a $1 \times B$ matrix with entries in $\mathcal{M}$, where B is some finite index set. The entries of ϕ are linear dependent over $\mathcal{C}$ if and only if the generalized Wronskians W_Y^ϕ of ϕ for all Young-like sets Y equal zero.*

For the proofs of these results, please refer to [10], specifically the proofs of Theorem 3.1 and Lemma 3.3 there. As noted in the remarks at the end of section 3 of that article, it can be seen that those proofs readily apply within our current context. (While the generalized Wronskians considered in [10] use lexicographical ordering instead of graded lexicographical ordering on rows, this only affects their sign. And the proof of Lemma 3.2 in [10] applies with any monomial ordering.)

3. Differential Rationality Criteria

Our objective in this section is to develop purely differential criteria for rationality in a multivariable differential field $\mathcal{M}$.

To begin, assume that $\xi_1, \xi_2, \ldots, \xi_m$ denote elements of $\mathcal{M}$ dual to the derivations $D_1, D_2, \ldots, D_m$, namely $D_j \xi_k = \delta_{jk}$. When $\operatorname{char} \mathcal{M} = 0$, this duality assumption implies (using Theorem 2.2 for instance) that any collection of monomials of ξ are linearly independent over $\mathcal{C}$, thus $\xi_1, \xi_2, \ldots, \xi_m$ are algebraically independent over $\mathcal{C}$. But when $\operatorname{char} \mathcal{M}$ equals a prime p, ξ_j^p belongs to $\mathcal{C}$ for each j. In this case, the maximal collection of monomials of ξ forming a linear independent set over $\mathcal{C}$ consists of those with exponents in $\mathcal{Q}_p := \{(a_1, a_2, \ldots, a_m) \in \mathbb{N}^m \mid a_j < p \text{ for all } j\}$. So it is suitable to define $\mathcal{Q}_0 = \mathbb{N}^m$, so that $\mathcal{Q}_{\operatorname{char} \mathcal{M}}$ serves as the *canonical exponent set*.

To say that $F \in \mathcal{M}$ is rational with respect to $\xi_1, \xi_2, \ldots, \xi_m$ simply means that there exist finite subsets $A, B \subseteq \mathcal{Q}_{\operatorname{char} \mathcal{M}}$ and coefficients $a_\alpha, b_\beta \in \mathcal{C}$ for $\alpha \in A$ and $\beta \in B$ with at least one $b_\beta \neq 0$ such that

$$F = \frac{\sum_{\alpha \in A} a_\alpha \xi^\alpha}{\sum_{\beta \in B} b_\beta \xi^\beta}. \tag{7}$$

It is a basic but instrumental observation that the question of rationality can be translated into a question of linear dependence.

Proposition 3.1. *Let $F \in \mathcal{M}$ and A, B be subsets of $\mathcal{Q}_{\operatorname{char} \mathcal{M}}$. F is rational with respect to $\xi_1, \xi_2, \ldots, \xi_m$ with the numerator and denominator having monomial terms with exponents in A and B, respectively, if and only if the collection $\{\xi^\alpha \mid \alpha \in A\} \cup \{\xi^\beta F \mid \beta \in B\}$ is linearly dependent over $\mathcal{C}$.*

Proof. Equation (7) readily implies the dependence relation

$$\sum_{\alpha \in A} a_\alpha \xi^\alpha - \sum_{\beta \in B} b_\beta \xi^\beta F = 0. \tag{8}$$

The converse is also true, with linear independence of the monomials with exponents in $\mathcal{Q}_{\operatorname{char} \mathcal{M}}$ giving that at least one b_β is non-zero and so $\sum_{\beta \in B} b_\beta \xi^\beta \neq 0$. □

Given finite subsets $A, B \subset \mathcal{Q}_{\operatorname{char} \mathcal{M}}$, let σ denote the $1 \times A$ row matrix whose entries are the monomials of ξ with exponents in A, i.e. entry-wise $\sigma^\alpha = \xi^\alpha$, and let τ denote the $1 \times B$ row matrix defined entry-wise by $\tau^\beta = \xi^\beta F$. Let ϕ be the concatenation of σ and τ, i.e. $[\sigma \,|\, \tau]$. So ϕ is a row matrix whose entries list the elements $\{\xi^\alpha \mid \alpha \in A\} \cup \{\xi^\beta F \mid \beta \in B\}$.

We adopt the convention of treating $D^\gamma F$ as zero whenever $\gamma \notin \mathbb{N}^m$. For $C \subseteq \mathbb{N}^m$, define $R_C^B[F]$ to be the $C \times B$ matrix defined entrywise via $(R_C^B[F])_\gamma^\beta = \frac{\gamma!}{(\gamma-\beta)!} D^{\gamma-\beta} F$.

The main result of this section can be expressed as follows.

Theorem 3.2. *Let $F \in \mathcal{M}$, and assume that A and B are finite Young-like subsets in $\mathcal{Q}_{\operatorname{char} \mathcal{M}}$. The following statements are equivalent.*

(1) *F is rational with respect to $\xi_1, \xi_2, \ldots, \xi_m$ with numerator and denominator having monomial exponents in A and B, respectively.*

(2) *The matrix* $R^B_{\mathbb{N}^m\backslash A}[F]$ *has rank less than* $|B|$ *over* $\mathcal{M}$.

(3) *For each set* $C \subseteq \mathbb{N}^m\backslash A$ *such that* $C \cup A$ *is Young-like and* $|C| = |A|$, *it holds that* $\det R^B_C[F] = 0$.

(4) *For each set* $C \subseteq \mathbb{N}^m\backslash A$ *such that* $C\cup A$ *is Young-like and* $|C\cap(B'+\mathbb{N}^m)| \geq |B'|$ *for all* $B' \subseteq B$, *it holds that* $\det R^B_C[F] = 0$.

Notes: 1) For B finite, saying that $|C \cap (B' + \mathbb{N}^m)| \geq |B'|$ for all $B' \subseteq B$ is equivalent to saying that there exists a bijective map $g : B \to C$ such that $g(\beta) \geq \beta$ for all $\beta \in B$.

2) For the case of zero characteristic, one elegant normalization of $R^B_C[F]$ is to consider the $C \times B$ matrix $T^B_C[F]$ defined entrywise via $(T^B_C[F])^\beta_\gamma = \frac{1}{(\gamma-\beta)!}F_{\gamma-\beta}$. Since each row $T^B_{\{\gamma\}}[F]$ equals $\frac{1}{\gamma!}R^B_{\{\gamma\}}[F]$, Theorem 3.2 holds with each $R^B_C[F]$ replaced by $T^B_C[F]$.

3) When only a single derivation is being considered, i.e., $m = 1$, then conditions 3 and 4 involve only one determinant. This case corresponds to the generalized Schwarzian differential expressions introduced in [13].

We remark that this theorem is purely differential in nature. No evaluations or integrations are involved. In particular, both conditions 3 and 4 reveal that rationality with bounds on degree or with any finite prescribed exponent sets A and B is equivalent to the satisfaction of a finite collection of (partial) differential equations.

Let $I_k = \{\alpha \in \mathbb{N}^m \mid |\alpha| \leq k\}$. The case $A = \mathcal{Q}_{\mathrm{char}\,\mathcal{M}} \cap I_M$ and $B = \mathcal{Q}_{\mathrm{char}\,\mathcal{M}} \cap I_N$ corresponds to testing for rationality with degree bounds of M and N on the numerator and denominator, respectively. So we may consider the following corollary.

Corollary 3.3. *Let* $F \in \mathcal{M}$, *and let* M, N *be nonnegative integers. Then* F *is rational with respect to* $\xi_1, \xi_2, \ldots, \xi_m$ *with numerator having degree at most* M *and numerator having degree at most* N *if and only if for each set* $C \subseteq I_{M+N+1}\backslash(\mathcal{Q}_{\mathrm{char}\,\mathcal{M}}\cap I_M)$ *for which* $C\cup(\mathcal{Q}_{\mathrm{char}\,\mathcal{M}}\cap I_M)$ *is Young-like,* $|C| = |\mathcal{Q}_{\mathrm{char}\,\mathcal{M}}\cap I_N|$, *and* $|C\backslash I_{M+N}| \leq 1$, *it holds that* $\det R^{I_N}_C[F] = 0$. *(So the differential equations that occur have order at most* $M+N+1$ *and those differential equations having order equal to* $M+N+1$ *are quasi-linear with only one partial derivative of order* $M + N + 1$ *appearing.)*

Proof. Set $A = \mathcal{Q}_{\mathrm{char}\,\mathcal{M}} \cap I_M$ and $B = \mathcal{Q}_{\mathrm{char}\,\mathcal{M}} \cap I_N$. Using condition 4 of Theorem 3.2, it suffices to show that if (i) $C\backslash I_{M+N+1} \neq \emptyset$ or (ii) $|C\backslash I_{M+N}| > 1$ then there exists a $B' \subseteq \mathcal{Q}_{\mathrm{char}\,\mathcal{M}} \cap I_N$ such that $|C'| < |B'|$ where $C' = C \cap (B' + \mathbb{N}^m)$. So we consider these two cases separately.

Case 1: Suppose that $C\backslash I_{M+N+1} \neq \emptyset$. Since $C \cup (\mathcal{Q}_{\mathrm{char}\,\mathcal{M}} \cap I_M)$ is Young-like, there exists a $\delta \in C$ so that $|\delta| = M + N + 2$. Let $\mathfrak{B} = \{\beta \in \mathbb{N}^m \mid \beta \leq \delta, |\beta| \leq N\}$ and $\mathfrak{C} = \{\gamma \in \mathbb{N}^m \mid \gamma \leq \delta, |\gamma| \geq M + 1\} \subseteq C$. Using the map $\gamma \mapsto \delta - \gamma$ we derive that $|\mathfrak{C}| = |\{\beta \in \mathbb{N}^m \mid \beta \leq \delta, |\beta| \leq N+1\}| > |\mathfrak{B}|$. Set $B' = B \backslash \mathfrak{B}$, and note that $C' \subseteq C \backslash \mathfrak{C}$. Thus $|C'| \leq |C| - |\mathfrak{C}| < |B| - |\mathfrak{B}| \leq |B'|$.

Case 2: Suppose that there exists two distinct $\delta_1, \delta_2 \in C$ such that $|\delta_1| = |\delta_2| = M + N + 1$. Define δ' to be the entry-wise minimum of δ_1 and δ_2, and note that $|\delta'| \leq M + N$. Let $\mathfrak{C}_1 = \{\gamma \in \mathbb{N}^m \mid \gamma \leq \delta_1, |\gamma| \geq M+1\}$, $\mathfrak{C}_2 = \{\gamma \in \mathbb{N}^m \mid \gamma \leq \delta_2, |\gamma| \geq M+1\}$, and $\mathfrak{C}_1 \cap \mathfrak{C}_2 = \{\gamma \in \mathbb{N}^m \mid \gamma \leq \delta', |\gamma| \geq M+1\}$, all of which are subsets of C. Also let $\mathfrak{B}_1 = \{\beta \in \mathbb{N}^m \mid \beta \leq \delta_1, |\beta| \leq N\}$, $\mathfrak{B}_2 = \{\beta \in \mathbb{N}^m \mid \beta \leq \delta_2, |\beta| \leq N\}$, and $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \{\beta \in \mathbb{N}^m \mid \beta \leq \delta', |\beta| \leq N\}$. By arguments similar to those in the previous case, one obtains that $|\mathfrak{C}_1| = |\mathfrak{B}_1|$, $|\mathfrak{C}_2| = |\mathfrak{B}_2|$, and $|\mathfrak{C}_1 \cap \mathfrak{C}_2| < |\mathfrak{B}_1 \cap \mathfrak{B}_2|$. Consequentially, $|\mathfrak{C}_1 \cup \mathfrak{C}_2| > |\mathfrak{B}_1 \cup \mathfrak{B}_2|$. So set $B' = B \backslash (\mathfrak{B}_1 \cup \mathfrak{B}_2)$, which implies $C' \subseteq C \backslash (\mathfrak{C}_1 \cup \mathfrak{C}_2)$ and so $|C'| < |B'|$.

□

Further pruning or reduction of the characterizing collection of differential equations arising from condition 4 of Theorem 3.2 (or from the above corollary) is possible. For instance, choices of C where there exist proper nontrivial subsets B' of B such that $|C'| = |B'|$ for $C' = C \cap (B' + \mathbb{N}^m)$ automatically exhibit the factorization

$$\det R_C^B[F] = \pm \det R_{C'}^{B'}[F] \det R_{C \backslash C'}^{B \backslash B'}[F]. \tag{9}$$

And one can often reason from suitable collections of these that only certain factors need be retained.

One more elegant means to accomplish part of the pruning obtainable by above is to include the relations from Theorem 3.2 for lesser values of m and remove the relations divisible by these lower degree relations. (For instance, consider Theorem 3.2 with any proper, non-trivial subset $\xi_{j_1}, \xi_{j_2}, \ldots, \xi_{j_\ell}$ (with complement $\xi_{k_1}, \xi_{k_2}, \ldots, \xi_{k_{m-\ell}}$) replacing A and B with the slices (which are also projections due to Young-likeness) $A'' = A \cap \{a_{k_1} = a_{k_2} = \cdots = a_{k_{m-\ell}} = 0\}$ and $B'' = B \cap \{b_{k_1} = b_{k_2} = \cdots = b_{k_{m-\ell}} = 0\}$.) Following this, further differential algebra reduction may still be helpful. We will illustrate this in Example 5 and Example 6.

In the case of prime characteristic p, one can derive that $\mathcal{C}(\xi_1, \xi_2, \ldots, \xi_m) = \mathcal{C}[\xi_1, \xi_2, \ldots, \xi_m]$ using standard field theory since $\xi_1, \xi_2, \ldots, \xi_m$ are algebraic over $\mathcal{C}$. But the differential criteria of Theorem 3.2 also provides a proof.

Proposition 3.4. *Assume* $\operatorname{char} \mathcal{M} = p$ *and* $F \in \mathcal{M}$. *Then* F *is rational with respect to* $\xi_1, \xi_2, \ldots, \xi_m$ *if and only if* F *is a polynomial with respect to* $\xi_1, \xi_2, \ldots, \xi_m$.

Proof. The reverse direction is clear. Assume that F is rational with respect to $\xi_1, \xi_2, \ldots, \xi_m$ (meaning with $A = B = \mathcal{Q}_p$). It suffices to show that $D_j^p F = 0$ for all j, as this shows that F is a polynomial using Theorem 3.2 with $A = \mathcal{Q}_p$ and $B = \{0\}$.

Now fix a value of j, let C equal the translate $pe_j + \mathcal{Q}_p$, and let let $B = \mathcal{Q}_p$. We claim that for $\gamma = (c_1, c_2, \ldots, c_m) \in C$ and $\beta = (b_1, b_2, \ldots, b_m) \in B$ with $\gamma \prec \beta + pe_j$ it follows that $\left(R_C^B[F]\right)_\gamma^\beta = 0$. This is clear when $c_k < b_k$ for $k \neq j$ seeing as $\gamma - \beta \notin \mathbb{N}^m$. Otherwise $c_j < b_j + p$, in which case $\binom{c_j}{b_j} b_j!$ (and thus $\binom{\gamma}{\beta}\beta!$) is divisible by p.

Therefore $R_C^B[F]$ is lower-triangular with diagonal entries $\left(R_C^B\right)_{\beta+pe_j}^\beta = \beta! D_j^p F$. So the rationality criteria implies that $0 = \det R_C^B[F] = \left(\prod_{\beta \in \mathcal{Q}_p} \beta!\right) (D_j^p F)^{p^m}$. Thus $D_j^p F = 0$.

□

So when $\operatorname{char} \mathcal{M} = p$, the collection of rational functions is a $\mathcal{C}$ vector space of dimension $|\mathcal{Q}_p| = p^m$. One consequence of this and Proposition 3.1 is the following corollary.

Corollary 3.5. *Let A and B be any subsets of $\mathcal{Q}_p$ such that $|A| + |B| > p^m$. Any F in $\mathcal{M}$ that is rational (thus polynomial) with respect to $\xi_1, \xi_2, \ldots, \xi_m$ can be expressed using a numerator and denominator having the exponents of their monomial terms confined to the sets A and B, respectively.*

To illustrate Theorem 3.2 and some of these related remarks, we mention the following examples. (For notational compactness, let $F_\gamma = D^\gamma F$.)

Example 5. Consider the two variable analog of linear fractional functions, which corresponds to setting $m = 2$, $M = 1$, and $N = 1$ in Corollary 3.3. Only the rows $\psi_\gamma = R_{\{\gamma\}}^{I_1}[F]$ where γ has order two or three are relevant. These rows are as follows.

$$
\begin{aligned}
\psi_{(0,2)} &= \begin{bmatrix} F_{(0,2)} & 2F_{(0,1)} & 0 \end{bmatrix} &\qquad \psi_{(0,3)} &= \begin{bmatrix} F_{(0,3)} & 3F_{(0,2)} & 0 \end{bmatrix} \\
\psi_{(1,1)} &= \begin{bmatrix} F_{(1,1)} & F_{(1,0)} & F_{(0,1)} \end{bmatrix} & \psi_{(1,2)} &= \begin{bmatrix} F_{(1,2)} & 2F_{(1,1)} & F_{(0,2)} \end{bmatrix} \\
\psi_{(2,0)} &= \begin{bmatrix} F_{(2,0)} & 0 & 2F_{(1,0)} \end{bmatrix} & \psi_{(2,1)} &= \begin{bmatrix} F_{(2,1)} & F_{(2,0)} & 2F_{(1,1)} \end{bmatrix} \\
& & \psi_{(3,0)} &= \begin{bmatrix} F_{(3,0)} & 0 & 3F_{(2,0)} \end{bmatrix}
\end{aligned}
$$

We point out that

$$
\det R_{\{(0,2),(1,1),(0,3)\}}^{\{(0,0),(0,1),(1,0)\}}[F] = -F_{(0,1)} \left(3F_{(0,2)}^2 - 2F_{(0,1)}F_{(0,3)}\right) \tag{10}
$$

and

$$
\det R_{\{(0,2),(2,0),(0,3)\}}^{\{(0,0),(0,1),(1,0)\}}[F] = -F_{(1,0)} \left(3F_{(0,2)}^2 - 2F_{(0,1)}F_{(0,3)}\right). \tag{11}
$$

Since the complete vanishing of $F_{(0,1)}$ will imply the vanishing of $\det R^{\{(0,0),(0,1)\}}_{\{(0,2),(0,3)\}}[F] = (3F^2_{(0,2)} - 2F_{(0,1)}F_{(0,3)})$, we may replace the vanishing of both expressions above with simply the vanishing of the latter factor. (Alternatively, one may note that the vanishing of $3F^2_{(0,2)} - 2F_{(0,1)}F_{(0,3)}$ is necessary as it is equivalent to F being linear fractional with respect to ξ_2. And its vanishing clearly guarantees the vanishing of both (10) and (11).) Employing Corollary 3.3, rationality with the parameters $M = 1$, $N = 1$ and $m = 2$ could be expressed as a system of seven partial differential equations. But using the observation above, this can be easily trimmed to the following equivalent system of five partial differential equations.

$$2F^2_{(0,1)}F_{(2,0)} - 4F_{(0,1)}F_{(1,0)}F_{(1,1)} + 2F^2_{(1,0)}F_{(0,2)} = 0 \tag{12}$$

$$2F_{(0,1)}F_{(0,3)} - 3F^2_{(0,2)} = 0 \tag{13}$$

$$2F^2_{(0,1)}F_{(1,2)} - 4F_{(0,1)}F_{(0,2)}F_{(1,1)} + F_{(1,0)}F^2_{(0,2)} = 0 \tag{14}$$

$$2F^2_{(1,0)}F_{(2,1)} + F_{(0,1)}F^2_{(2,0)} - 4F_{(1,0)}F_{(1,1)}F_{(2,0)} = 0 \tag{15}$$

$$2F_{(1,0)}F_{(3,0)} - 3F^2_{(2,0)} = 0 \tag{16}$$

Even further reduction is possible. (After all, these equations are not autoreduced.) Via calculations using a computer algebra system with a differential algebra abilities such as MAPLE (or via very lengthy calculations by hand), one can show that the radical differential ideal generated by equations (12) and (13) would contain equations (14) and (15) when $\operatorname{char} \mathcal{M} = 0$. In fact, by a careful walkthrough of delta-polynomial calculations, one can show that such also holds whenever $\operatorname{char} \mathcal{M}$ is not two or three. In characteristic two, the entire set of five can be replaced with $F_{(2,0)} = 0$ and $F_{(0,2)} = 0$, which is not surprising in light of Corollary 3.5. In characteristic three, equations (13) and (16) reduce to $F_{(0,3)} = 0$ and $F_{(3,0)} = 0$. Thus F may thus be expressed as the polynomial $\sum_{\alpha \in \mathcal{Q}_3} f_\alpha \xi^\alpha$. From here, one can use purely algebraic techniques to show that such a polynomial satisfying (12) will satisfy both (14) and (15). So equations (12), (13), and (16) suffice by themselves to characterize rationality in this case.

Example 6. Consider the case $m = 2$, $A = B = \{(0,0),(0,1),(1,0),(1,1)\}$. The following rows $\psi_\gamma = R^B_{\{\gamma\}}[F]$ are the only ones needed within condition 3.

$$\psi_{(0,2)} = \begin{bmatrix} F_{(0,2)} & 2F_{(0,1)} & 0 & 0 \end{bmatrix} \quad \psi_{(1,2)} = \begin{bmatrix} F_{(1,2)} & 2F_{(1,1)} & F_{(0,2)} & 2F_{(0,1)} \end{bmatrix}$$
$$\psi_{(0,3)} = \begin{bmatrix} F_{(0,3)} & 3F_{(0,2)} & 0 & 0 \end{bmatrix} \quad \psi_{(1,3)} = \begin{bmatrix} F_{(1,3)} & 3F_{(1,2)} & F_{(0,3)} & 3F_{(0,2)} \end{bmatrix}$$
$$\psi_{(2,0)} = \begin{bmatrix} F_{(2,0)} & 0 & 2F_{(1,0)} & 0 \end{bmatrix} \quad \psi_{(2,1)} = \begin{bmatrix} F_{(2,1)} & F_{(2,0)} & 2F_{(1,1)} & 2F_{(1,0)} \end{bmatrix}$$
$$\psi_{(3,0)} = \begin{bmatrix} F_{(3,0)} & 0 & 3F_{(2,0)} & 0 \end{bmatrix} \quad \psi_{(3,1)} = \begin{bmatrix} F_{(3,1)} & F_{(3,0)} & 3F_{(2,1)} & 3F_{(2,0)} \end{bmatrix}$$

There exist seven choices of C in condition 4 in this case, three of which yield

$$\det R^{\{(0,0),(0,1),(1,0),(1,1)\}}_{\{(0,2),(0,3),(1,2),(1,3)\}}[F] = \left(3F^2_{(0,2)} - 2F_{(0,1)}F_{(0,3)}\right)^2 = \left(\det R^{\{(0,0),(0,1)\}}_{\{(0,2),(0,3)\}}[F]\right)^2, \tag{17}$$

$$\det R^{\{(0,0),(0,1),(1,0),(1,1)\}}_{\{(2,0),(2,1),(3,0),(3,1)\}}[F] = \left(3F^2_{(2,0)} - 2F_{(1,0)}F_{(3,0)}\right)^2 = \left(\det R^{\{(0,0),(1,0)\}}_{\{(2,0),(3,0)\}}[F]\right)^2, \tag{18}$$

and

$$\begin{aligned} &\det R^{\{(0,0),(0,1),(1,0),(1,1)\}}_{\{(0,2),(2,0),(1,2),(2,1)\}}[F] \\ &= -8F^2_{(0,1)}F_{(1,0)}F_{(2,1)} + 8F_{(0,1)}F^2_{(1,0)}F_{(1,2)} + 8F^2_{(0,1)}F_{(1,1)}F_{(2,0)} - 8F^2_{(1,0)}F_{(0,2)}F_{(1,1)}. \end{aligned} \tag{19}$$

Of the remaining four choices for C, each $\det R^B_C$ possesses either $\det R^{\{(0,0),(0,1)\}}_{\{(0,2),(0,3)\}}[F]$ or $\det R^{\{(0,0),(1,0)\}}_{\{(2,0),(3,0)\}}[F]$ as a factor. One can show, using calculations carried out by a symbolic differential algebra system, that (19) belongs to the radical differential ideal generated by $\left(3F^2_{(0,2)} - 2F_{(0,1)}F_{(0,3)}\right)$ and $\left(3F^2_{(2,0)} - 3F_{(1,0)}F_{(3,0)}\right)$ when characteristic is not two or three. In characteristic two, this holds vacuously and $F_{(2,0)} = 0$ and $F_{(0,2)} = 0$ are the defining equations. In characteristic three, the equations (17) and (18) simply yield $F_{(3,0)} = 0$ and $F_{(0,3)} = 0$ which do not induce (19). So characterization of rationality with the given A and B is equivalent to the pair of equations (13) and (16), barring the case $\operatorname{char} \mathcal{M} = 3$.

Note: In both Example 5 and Example 6, it is necessary that F be rational with degree 1 with respect to ξ_1 and ξ_2 separately, which implies (13) and (16) by simply using the case of a single derivation. In Example 6 we see that such is sufficient (as one might desire), except for the unusual case $\operatorname{char} \mathcal{M} = 3$.

Example 7. Consider F in some differential field extension $\mathcal{M}$ of $\mathbb{Z}_3(\xi_1, \xi_2)$ with the derivation $D_1 = \frac{\partial}{\partial \xi_1}$. Let $A = B = \{0, 1\}$. Observe that Corollary 3.5 applies. Thus rationality with these prescribed exponent sets is equivalent to being a polynomial with respect to ξ_1 over $\mathcal{C}$, meaning that $D_1^3 F = 0$ is an equivalent condition. For instance, consider $F = \xi_1^2 + \xi_1\xi_2$, which can be specifically re-expressed as $\frac{(\xi_1^3) - (\xi_2^2)\xi_1}{(-\xi_2) + \xi_1}$. Also this serves as an interesting example from the point of view of Example 6. Notice that F is rational (in fact polynomial) with degree one with respect to just ξ_2. However F does not satisfy (19) (seeing as all but the third term vanish), and thus it is not rational with the prescribed exponent sets of Example 6.

In preparation for a proof of Theorem 3.2, we first present a supporting lemma. Let V_B denote the $B \times B$ matrix with entries given by $(V_B)^\beta_\kappa = \binom{\beta}{\kappa}\xi^{\beta-\kappa}$. Note that V_B is upper triangular with ones on the diagonal, thus it is invertible. For C a subset of $\mathbb{N}^m$, let U_C denote the $C \times C$ diagonal matrix where the diagonal entries are defined by $(U_C)^\gamma_\gamma = \gamma!$. Notice that $R^B_C[F] = U_C T^B_C[F]$. For $\gamma \in \mathbb{N}^m$ and $\beta \in \mathcal{Q}_{\mathrm{char}\,\mathcal{M}}$, observe that

$$(L^\tau_C)^\beta_\gamma = D^\gamma(\xi^\beta F) = \sum_{\kappa:0\le\kappa\le\gamma}\binom{\gamma}{\kappa}D^\kappa\xi^\beta D^{\gamma-\kappa}F \tag{20}$$
$$= \sum_{\kappa:0\le\kappa\le\beta}\binom{\gamma}{\kappa}\kappa!\binom{\beta}{\kappa}\xi^{\beta-\kappa}D^{\gamma-\kappa}F = \sum_{\kappa:0\le\kappa\le\beta}(R^B_C[F])^\kappa_\gamma(V_B)^\beta_\kappa = (R^B_C[F]V_B)^\beta_\gamma,$$

as long as $\kappa \in B$ for $0 \le \kappa \le \beta$, which is guaranteed if B is Young-like. This proves the following.

Lemma 3.6. *For $F \in \mathcal{M}$, $C \subseteq \mathbb{N}^m$, and B a finite Young-like subset of $\mathcal{Q}_{\mathrm{char}\,\mathcal{M}}$ it holds that*

$$L^\tau_C = R^B_C[F]V_B = U_C T^B_C[F]V_B. \tag{21}$$

Proof (of Theorem 3.2): We define the following statements which will serve as intermediate stepping stones in a series of equivalent statements.

(5) The entries of ϕ are linearly dependent over $\mathbb{C}$.
(6) The set $\bigcap_{\gamma\in\mathbb{N}^m}\ker D^\gamma\phi$ contains a non-zero vector in $\mathcal{M}^{|A|+|B|}$.

($1 \iff 5$) Apply Proposition 3.1.

($5 \iff 6$) Apply Lemma 2.1.

($6 \iff 2$) Let $\alpha \in A$ and $\gamma \in \mathbb{N}^m$, and note that $D^\gamma\xi^\alpha = 0$ whenever $\gamma \not\le \alpha$ (thus when $\alpha \prec \gamma$ or $\gamma \notin A$). Also $D^\alpha\xi^\alpha = \alpha!$. So $P_{A,\mathbb{N}^m\setminus A}L^\phi_{\mathbb{N}^m}$ has the upper triangular block-wise form

$$P_{A,\mathbb{N}^m\setminus A}L^\phi_{\mathbb{N}^m} = \begin{bmatrix} L^\sigma_A & L^\tau_A \\ 0 & L^\tau_{\mathbb{N}^m\setminus A}\end{bmatrix}. \tag{22}$$

and L^σ_A is an invertible, upper triangular matrix. So the dimension of the null space of $L^\phi_{\mathbb{N}^m}$ equals the dimension of the null space of $L^\tau_{\mathbb{N}^m\setminus A}$. Then, by using Lemma 3.6, this also equals the dimension of the null space of $R^B_{\mathbb{N}^m\setminus A}[F]$. Thus condition 6 is equivalent to condition 2.

($5 \iff 3$) By Theorem 2.2 it holds that condition 5 is equivalent to saying $W^\phi_Y = 0$ for all Young-like sets $Y \subseteq \mathbb{N}^m$ of size $|A| + |B|$. By considering the rows of (22), if $A \not\subseteq Y$ then L^σ_Y has rank less than $|A|$ and so W^ϕ_Y is zero, regardless of

F. So condition 5 is equivalent to saying that $W_Y^\phi = 0$ for all Young-like sets Y that contain A.

So assume $Y = A \cup C$ where $C = Y \backslash A$. Thus in block-wise form

$$P_{A,C} L_Y^\phi = \begin{bmatrix} L_A^\sigma & L_A^\tau \\ 0 & L_C^\tau \end{bmatrix}. \tag{23}$$

So $W_Y^\phi = \pm \det L_A^\sigma \det L_C^\tau$. Note that $\det L_A^\sigma = \prod_{\gamma \in A} \gamma! \neq 0$, seeing as $A \subseteq \mathcal{Q}_{\mathrm{char}\,\mathcal{M}}$. Also Lemma 3.6 gives that $\det L_C^\tau = \det R_C^B[F]$. So the vanishing of all generalized Wronskians W_Y^ϕ for Young-like sets Y containing A is equivalent to condition 3.

($3 \iff 4$) The forward direction is clear. For the reverse direction, let $C \subseteq \mathbb{N}^m \backslash A$ such that $|C| = |B|$ and $C \cup A$ is Young-like. It suffices to show that if there exists a $B' \subseteq B$ such that $|C'| < |B'|$, where $C' = C \cap (B' + \mathbb{N}^m)$, then $\det R_C^B[F] = 0$ regardless of F.

For $\gamma \in C \backslash C'$ and $\beta \in B'$ notice that $\gamma - \beta \notin \mathbb{N}^m$, thus the matrix $R_{C\backslash C'}^{B'}[F]$ is entirely zero, which implies that $\mathrm{rnk}\, R_{C\backslash C'}^B \leq |B| - |B'| < |C| - |C'|$. So $\det R_C^B[F] = 0$, independent of F.

□

Remark: The coefficients a_α and b_β for F in (7) can be determined using any reduced basis of the null space of $L_{\mathbb{N}^m}^\phi$, owing to the use of Lemma 2.1 in the proof of Theorem 3.2. Moreover the block-wise form given in (22) shows that the coefficients b_β can be determined using any reduced basis to the null space of $L_{\mathbb{N}^m \backslash A}^\tau$, which is $L_{\mathcal{Y}\backslash A}^\tau$ where $\mathcal{Y}$ is the union of all Young-like sets of size $|A| + |B|$ that contain A. Then the coefficients a_α can be determined using backwards substitution or by using (8). So one can construct differential expressions for the coefficients a_α and b_β that are valid on a Zariski open set within the Zariski closed set given by the rationality criteria.

For instance, consider Example 5 discussed after Theorem 3.2, and assume that neither $F_{(0,2)}$ nor $F_{(0,1)}$ are identically zero. Then $\psi_{(0,2)}$ and $\psi_{(1,1)}$ are linearly independent over $\mathcal{M}$ and so $D^{(0,2)}\tau$ and $D^{(1,1)}\tau$ behave likewise. Then a reduced basis for the row space of $L_{\mathbb{N}^m \backslash A}^\tau$ is given by $\begin{bmatrix} 1 & \frac{2D^{(0,1)}F}{D^{(0,2)}F} + \xi_2 + \xi_1 \left(\frac{-D^{(1,0)}F}{D^{(0,1)}F} + \frac{2D^{(1,1)}F}{D^{(0,2)}F} \right) & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \frac{D^{(1,0)}F}{D^{(0,1)}F} - \frac{2D^{(1,1)}F}{D^{(0,2)}F} & 1 \end{bmatrix}$, when the rationality criteria is satisfied. Thus we may set $b_{(0,0)} = \frac{-2D^{(0,1)}F}{D^{(0,2)}F} - \xi_2 + \xi_1 \left(\frac{D^{(1,0)}F}{D^{(0,1)}F} - \frac{2D^{(1,1)}F}{D^{(0,2)}F} \right)$, $b_{(0,1)} = 1$ and $b_{(1,0)} = -\frac{D^{(1,0)}F}{D^{(0,1)}F} + \frac{2D^{(1,1)}F}{D^{(0,2)}F}$. These expressions are not a priori constant, but they will produce constants when the rationality criteria holds with the given parameters.

4. Taylor Series Rationality Criteria

Another approach is to characterize the Taylor series of rational functions. Such has classical analytic motivations, but such can also phrased in a more general differential algebra setting with the introduction of evaluations.

We assume throughout this section that $\mathcal{M}$ has characteristic zero. Given an evaluation E, we define the Taylor series homomorphism $T_E : \mathrm{Dom}_E \to \mathcal{C}[[\xi_1, \xi_2, \ldots, \xi_m]]$ (or simply calling it T when E is understood) by

$$T_E(F) = \sum_{\alpha \in \mathbb{N}^m} \frac{1}{\alpha!} E(D^\alpha F) \xi^\alpha. \tag{24}$$

(As before, assume that $\xi_1, \xi_2, \ldots, \xi_m$ are dual to $D_1, D_2, \ldots, D_m$. Also assume that $E(\xi_j) = 0$ for each j, which can be achieved by replacing ξ_j with $\xi_j - E(\xi_j)$.) We say that Dom_E (or by association $\mathcal{M}$) is *Taylor-regular* with respect to E if T_E is injective. In this case, Dom_E is isomorphic to a differential subdomain of $\mathcal{C}[[\xi_1, \xi_2, \ldots, \xi_m]]$ with E being the evaluation $\sum_{\alpha \in \mathbb{N}^m} f_\alpha \xi^\alpha \mapsto f_0$ and $\mathcal{M}$ is isomorphic to a differential subfield of $\mathcal{C}((\xi_1, \xi_2, \ldots, \xi_m))$.

In our present context, it is sensible to assume the Taylor-regularity of Dom_E, so that a function is completely determined by its Taylor coefficients. If one removes the assumption of Taylor-regularity, then the results that following would serve to characterize when a function's abstract Taylor series $T(F)$ is rational.

The following is a basic Taylor series analog of Theorem 3.2.

Theorem 4.1. *Assume that* $\operatorname{char} \mathcal{M} = 0$ *and* $\mathcal{M}$ *has a evaluation* E *with Taylor-regular domain* Dom_E*. Let* $F \in \mathrm{Dom}_E$ *and let* A *and* B *be finite Young-like sets* $\mathbb{N}^m$*. The following are equivalent.*

1. *F is a rational with respect to $\xi_1, \xi_2, \ldots, \xi_m$ with numerator and denominator having monomial exponents in A and B, respectively.*
2. *The matrix $E(T^B_{\mathbb{N}^m \setminus A}[F])$ has rank less than $|B|$ over $\mathcal{C}$.*
3. *For each set $C \subseteq \mathbb{N}^m \setminus A$ such that $|C| = |B|$ and $|C \cap (B' + \mathbb{N}^m)| \geq |B'|$ for all $B' \subseteq B$, it holds that $E(\det T^B_C[F]) = 0$.*

Proof. We define the following statement, which we will show is an an additional equivalent statement.

(4) The set $\bigcap_{\gamma \in \mathbb{N}^m} \ker E(D^\gamma \phi)$ contains a non-zero vector in $\mathcal{C}^{|A|+|B|}$.

($1 \iff 4$) Suppose condition 1, then there exists a non-zero $\vec{v} \in \mathcal{C}^{|A|+|B|}$ such that $\phi \vec{v} = 0$. By applying D^γ and E to this equation, we obtain that $E(D^\gamma \phi) \vec{v} = 0$ for any $\gamma \in \mathbb{N}^m$. Thus condition 4 holds.

Conversely, assume that there is a non-zero $\vec{v} \in \mathcal{C}^{|A|+|B|}$ such that $E(D^\gamma \phi)\vec{v} = 0$ for any $\gamma \in \mathbb{N}^m$. Thus $E(D^\gamma(\phi\vec{v})) = 0$ for all γ, so Taylor regularity implies that $\phi\vec{v} = 0$. Thus condition 1 holds.

($4 \iff 2$) Applying the evaluation E to (22), we obtain that

$$P_{A,\mathbb{N}^m\setminus A} E(L^\phi_{\mathbb{N}^m}) = \begin{bmatrix} U_A & E(L^\tau_A) \\ 0 & E(L^\tau_{\mathbb{N}^m\setminus A}) \end{bmatrix}. \tag{25}$$

And applying E to the result of Lemma 3.6 gives that $E(L^\tau_{\mathbb{N}^m\setminus A}) = U_{\mathbb{N}^m\setminus A} E(T^B_{\mathbb{N}^m\setminus A}[F])$. It follows that $E(T^B_{\mathbb{N}^m\setminus A}[F])$ has a non-trivial null space exactly when the same is true of $E(L^\phi_{\mathbb{N}^m})$. Therefore conditions 4 and 2 are equivalent.

($2 \iff 3$) Condition 2 is equivalent to saying that every collection of $|B|$ rows from $E(T^B_{\mathbb{N}^m\setminus A}[F])$ has less than full rank, namely $E(\det T^B_C[F]) = 0$ for all $C \subseteq \mathbb{N}^m\setminus A$. To see that this is equivalent to 3, it suffices to show that $E(\det T^B_C[F]) = 0$ if there exists a $B' \subseteq B$ such that $|C \cap (B' + \mathbb{N}^m)| < |B'|$. In this case, it was shown in the proof of Theorem 3.2 that $\det T^B_B[F] = 0$, so $E(\det T^B_C[F]) = 0$ also follows. □

It follows from Theorem 4.1 that rationality of $F = \sum_{\alpha\in\mathbb{N}^m} f_\alpha \xi^\alpha$ with finite exponent sets A and B is equivalent to a set of algebraic relations on $\{f_\alpha\}$. So this defines an algebraic variety (or algebraic set) within the infinite dimensional vector space of formal power series. (But without reduction, the relations Theorem 4.1 would not directly define such in the sense of a scheme, seeing as these relations all have degree $|B|$ but relations of lesser degree exist when $m > 1$ and $|B| > 1$.)

As with Theorem 3.2, the relations produced by condition 3 can be pruned or reduced. For instance, applying E to equation (9) shows that a number of relations factor. As before, one elegant first sweep is to incorporate the relations produced by 3 for proper subcollections of $\xi_1, \xi_2, \ldots, \xi_m$ (with the corresponding projections of the permitted exponent sets A and B) and then remove those relations properly divisible by these lower degree relations.

Another Taylor series result would be to simply note that the vanishing of the Taylor coefficients of the differential equations occuring in Theorem 3.2 (or their pruned and reduced counterparts) also produces a characterization of rationality with prescribed exponent sets, if we assume Taylor regularity. One can show that such relations are linear combinations of the relations produced in condition 3.

(In fact, if one sought an analogous result for prime characteristic, one could replace the Taylor series (24) with simply $F = \sum_{\alpha\in\mathcal{Q}_{\mathrm{char}\,\mathcal{M}}} f_\alpha \xi^\alpha$ in light of Proposition 3.4. And one can readily convert the differential relations of condition 4 of Theorem 3.2 into a set of algebraic relations on the coefficients f_α.)

Theorem 4.4 will give another Taylor series approach, one based on rationality radial one-dimensional slices.

4.1. **The single variable case,** $m = 1$**.** In preparation for the approach of radial slices, it will be helpful to recall and develop some specifics concerning the case $m = 1$.

First we refresh and specialize our notation for this subsection as follows. Let $\tilde{\mathcal{M}}$ denote a differential field with one stated derivation $\tilde{D}$ and an evaluation $\tilde{E}$ having a Taylor-regular domain $\mathrm{Dom}_{\tilde{E}}$. Assume that t is an element of $\tilde{\mathcal{M}}$ such that $\tilde{D}(t) = 1$ and $\tilde{E}(t) = 0$. So $\tilde{\mathcal{M}}$ corresponds to a differential subfield of $\mathcal{C}((t))$ where $\tilde{D} = \frac{\partial}{\partial t}$ and $\tilde{E} : \sum_{j=0}^{\infty} f_j t^j \mapsto f_0$. For $F = \sum_{j=0}^{\infty} f_j t^j \in \mathrm{Dom}_{\tilde{E}}$, define the normalized derivative $\tilde{F}_j = \frac{1}{j!} D^j F$ for $j \geq 0$ and $F_j = 0$ if $j < 0$. So $f_j = \tilde{E}(\tilde{F}_j)$. Consistent with prior notation, let

$$\tilde{T}^{\{0,1,\ldots,N\}}_{\{k_1,k_2,\ldots,k_\ell\}}[F] = \begin{vmatrix} \tilde{F}_{k_1} & \tilde{F}_{k_1-1} & \cdots & \tilde{F}_{k_1-N} \\ \tilde{F}_{k_2} & \tilde{F}_{k_2-1} & \cdots & \tilde{F}_{k_2-N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{F}_{k_\ell} & \tilde{F}_{k_\ell-1} & \cdots & \tilde{F}_{k_\ell-N} \end{vmatrix}. \tag{26}$$

We define $\mathcal{H}_{M,N}[F] = \tilde{E}(\det \tilde{T}^{\{0,1,\ldots,N\}}_{\{M+1,M+2,\ldots,M+N+1\}}[F])$, namely

$$\mathcal{H}_{M,N}[F] = \begin{vmatrix} f_{M+1} & f_M & \cdots & f_{M-N+1} \\ f_{M+2} & f_{M+1} & \cdots & f_{M-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{M+N+1} & f_{M+N} & \cdots & f_{M+1} \end{vmatrix}. \tag{27}$$

Theorem 4.2. *With the definitions above, the following are equivalent*

(1) *F is rational with respect to t with numerator and denominator having degrees at most M and N, respectively,*
(2) *$\tilde{E}(\det \tilde{T}^{\{0,1,\ldots,N\}}_{\{k_1,k_2,\ldots,k_{N+1}\}}[F]) = 0$ for any choices of $k_1, k_2, \ldots, k_{N+1} \geq M + 1$,*
(3) *$\mathcal{H}_{M+k,N+k}[F] = 0$ for $k \geq 0$, and*
(4) *$\mathcal{H}_{M+k,N}[F] = 0$ for $k \geq 0$.*

This result is fairly classical [3]. (pp.321-323). We present a proof below as certain components of this proof will prove useful later.

Proof. ($1 \iff 2$) This follows as a special case of the equivalence of conditions 1 and 3 in Theorem 4.1.

($1 \implies 3$) Assume that condition 1 holds. Then $\mathcal{H}_{k,N}[F] = 0$ holds for all $k \geq M$ as a result of condition 2. Also condition 1 automatically holds for all larger values of N. Thus we obtain that $\mathcal{H}_{k,\ell}[F] = 0$ for any $k \geq M$ and $\ell \geq N$. So condition 3 (as well as condition 4) clearly follow.

($3 \implies 4$) Sylvester's determinant identity applied to Toeplitz matrices shows that

$$\mathcal{H}_{M,N}[F]^2 - \mathcal{H}_{M-1,N}[F]\mathcal{H}_{M+1,N}[F] = \mathcal{H}_{M,N-1}[F]\mathcal{H}_{M,N+1}[F]. \tag{28}$$

Assume condition 3 holds. Then for any $k \geq 0$, $\mathcal{H}_{M+k,N+k}[F] = 0$ and $\mathcal{H}_{M+k+1,N+k+1}[F] = 0$ imply that $\mathcal{H}_{M+k+1,N+k}[F] = 0$. Repeating this argument in an inductive fashion reveals that $\mathcal{H}_{M+k+\ell,N+k}[F] = 0$ for $k, \ell \geq 0$. Setting $k = 0$ obtains condition 4.

(4 $\implies$ 2) Assume condition 4 holds. Holding M fixed, shrink N to be the minimal choice for which condition 4 remains valid. If $\mathcal{H}_{M-1+k,N-1}[F] = 0$ for at least one $k \geq 0$, then the determinant identity (28) can be used to show that $\mathcal{H}_{M+k,N-1}[F] = 0$ for all $k \geq 0$. So with N minimally chosen we have that (i) $N = 0$ or (ii) $\mathcal{H}_{M-1+k,N-1}[F] \neq 0$ for $k \geq 0$. In the case $N = 0$ then $f_{M+k} = 0$ for $k \geq 0$ and F is a polynomial of degree at most M. In the other case, since $\mathcal{H}_{M+k,N} = 0$ and $\mathcal{H}_{M+k,N-1}[F] \neq 0$ it follows that the first N rows used to define $\mathcal{H}_{M+k,N}[F]$ are linearly independent over $\mathcal{C}$ and their $\mathcal{C}$-span includes the $(N+1)$th row. So we may inductively conclude that each row $\begin{bmatrix} f_{M+1+k} & f_{M+k} & \cdots & f_{M-N+1+k} \end{bmatrix}$ for $k \geq 0$ is in the $\mathcal{C}$-span of $\left\{ \begin{bmatrix} f_{M+1+k} & f_{M+k} & \cdots & f_{M-N+1+k} \end{bmatrix} \right\}_{k=0}^{N-1}$. Therefore condition 2 and thus condition 1 hold, possibly with a smaller value of N. But this implies condition 1 and thus condition 2 for the original value of N.

□

Remarks:

1) We can construct explicit expressions in terms of Taylor coefficients for $a_0, \ldots a_M$, $b_0, \ldots, b_N$ in $\mathcal{C}$ so that $F = \frac{a_0 + a_1 t + \cdots + a_M t^M}{b_0 + b_1 t + \cdots + b_N t^N}$. Assume that condition 4 holds and either $N = 0$ or $\mathcal{H}_{M-1,N-1} \neq 0$, in which case these coefficients will be uniquely determined up to a constant multiple. Notably $\begin{bmatrix} b_0 & b_1 & \cdots & b_N \end{bmatrix}^{\mathrm{T}}$ is in the null space of $E(T^{\{0,1,\ldots,N\}}_{\{M+1,M+2,\ldots,M+N\}}[F])$, which will be one-dimensional with the given assumptions. Using Cramer's rule, it follows that $b_j = (-1)^j b_0 \frac{\det C_{\hat{j}}}{\det C_{\hat{0}}}$, where $C_{\hat{j}}$ equals $E(T^{\{0,1,\ldots,N\}}_{\{M+1,M+2,\ldots,M+N\}}[F])$ with the last row and $(j+1)$th column deleted. (Note that $\det C_{\hat{0}} = \mathcal{H}_{M-1,N-1}[F] \neq 0$, by assumption.) Setting $b_0 = \det C_{\hat{0}}$ we obtain that

$$b_j = (-1)^j \begin{vmatrix} f_{M+1} & \cdots & f_{M-j+2} & f_{M-j} & \cdots & f_{M-N+1} \\ f_{M+2} & \cdots & f_{M-j+3} & f_{M-j+1} & \cdots & f_{M-N+2} \\ \vdots & & \vdots & \vdots & & \vdots \\ f_{M+N} & \cdots & f_{M+N-j+1} & f_{M+N-j-1} & \cdots & f_M \end{vmatrix}. \tag{29}$$

By using the upper rows of (25) or, more simply, by considering $(a_0 + a_1 t + \cdots + a_M t^M) = F \cdot (b_0 + b_1 t + \cdots + b_N t^N)$, it follows that

$$a_j = -f_0 b_j - f_1 b_{j-1} - \cdots - f_j b_0, \tag{30}$$

taking b_j to be zero if $j > N$.

2) If we wish to construct a Taylor series satisfying any of the equivalent conditions of the theorem, it is worth noting that the coefficients f_j for $j \leq M - N$

are completely arbitrary, and that the coefficients f_j for $j \geq M+N+1$ are uniquely determined by the values of f_j for $M-N+1 \leq j \leq M+N$. The first part of this statement can be seen by observing that (27) is independent of f_j for $j < M-N+1$ or alternatively by noting that subtracting an arbitrary polynomial of degree $M-N$ from F does not affect the validity of condition 1. When $\mathcal{H}_{M-1,N-1}[F] \neq 0$, the second part can be seen inductively by considering $\tilde{E}(\tilde{S}_{M+1,\ldots,M+N,j}[F])$ for $j > M+N$. And when $\mathcal{H}_{M-1,N-1}[F] = 0$, one can simply reduce to a lesser value of N' for which either $\mathcal{H}_{M-1,N'-1}[F] \neq 0$ or $N' = 0$. In contrast, the values of f_j for $M-N+1 \leq j \leq M+N$ cannot be chosen completely arbitrarily (though choosing them so that $\mathcal{H}_{M-1,N-1}[F] = 0$ is one sufficient condition). Instead the possible choices of these f_j forms a constructible set, being the projection of an algebraic variety.

3) The collection of rational functions (with bounds given as in 1) has the structure of an algebraic variety in the space of sequences $(f_0, f_1, \ldots)$. We pause to mention that the relations produced by condition 1 are also interesting from the perspective of Groebner basis theory. (We note that much of the foundational theory of Groebner bases developed for finitely many indeterminates can be extended to polynomial rings with infinitely many indeterminates. The Groebner bases may be infinite, but they have cardinality no greater than the cardinality of the set of indeterminates. So these varieties present some interesting examples in the area of infinite dimensional algebraic geometry. For instance the case $M = 0$, $N = 1$ can be viewed as a generalization of the classic twisted cubic.)

To further elaborate on this point, define $G_1 = \{\tilde{E}(\det \tilde{T}^{\{0,1,\ldots,N\}}_{\{k_1,k_2,\ldots,k_{N+1}\}}[F]) \mid M < k_1 < k_2 < \cdots < k_{N+1}\}$ and $G_2 = \{\mathcal{H}_{M+k,N}[F] \mid k \geq 0\}$ as subsets of the polynomial ring $\mathbb{Q}[f_0, f_1, \ldots]$. Namely G_1 and G_2 correspond to the relations in conditions 2 and 4, respectively. With suitable changes in sign, G_1 forms a minimal Groebner basis of its generated ideal, under either (1) graded reverse lexicographical ordering with the ranking $f_0 < f_1 < \cdots$ or (2) graded lexicographical ordering (or lexicographical ordering as the generators are homogeneous) with the ranking $f_0 < f_1 < \cdots$, if $M - N - 1 \geq 0$. In particular the elements of G_1 have S-polynomials that reduce to zero modulo G_1. To outline why, we highlight the identity

$$f_{k_\ell-\ell+1}\tilde{E}(\det \tilde{T}_{k_1,k_2,\ldots\hat{k_\ell}\ldots,k_{N+2}}[F]) - f_{k_{\ell+1}-\ell+1}\tilde{E}(\det \tilde{T}_{k_1,k_2,\ldots,\widehat{k_{\ell+1}}\ldots k_{N+2}}[F]) = \sum_{\substack{j=1,\ldots,N+2 \\ j\neq \ell,\ell+1}} (-1)^{j-\ell+1} f_{k_j-\ell+1}\tilde{E}(\det \tilde{T}_{k_1,k_2,\ldots,\hat{k_j}\ldots k_{N+2}}[F]), \tag{31}$$

for $k_{N+2} > k_{N+1} > \cdots > k_1 \geq M+1$ and $1 \leq \ell \leq N+2$. (This follows as the matrix $T^{0,1,\ldots,N}_{k_1,k_2,\ldots k_{N+2}}[F]$ with the ℓth column repeated has determinant zero.) One can show that the leading terms of the $f_{k_j-\ell+1}\tilde{E}(\det \tilde{T}_{k_1,k_2,\ldots,\hat{k_j}\ldots k_{N+2}}[F])$ in the summation on the right are distinct, thus the specific S-polynomial on the left reduces to 0 modulo G_1. Then, by employing the techniques exposited in [2] Chapter 2, Section 9 (namely Propositions 4 and 10), we can conclude that all other S-polynomials reduce to 0. G_2 also forms a minimal Groebner basis for its generated ideal under graded reverse lexicographical ordering, seeing as the leading terms are relatively prime.

G_1 and G_2 generate different ideals but they share the same radical ideal. This follows from Theorem 4.2 (including its generality with respect to $\mathcal{C}$) and by applying an infinite dimensional version of Hilbert's Nullstellensatz [7]. (More specifically, this holds with any uncountable, algebraically closed field extension $\hat{\mathcal{C}}$ of $\mathbb{Q}$, but $\mathbb{Q}$ also inherits the result since $\langle \hat{\mathcal{C}}[f_0, f_1, \ldots]I\rangle \cap \mathbb{Q}[f_0, f_1, \ldots] = I$ for any ideal I in $\mathbb{Q}[f_0, f_1, \ldots]$.) However the corresponding elimination ideals $\langle G_1\rangle \cap \mathbb{Q}[f_0, f_1, \ldots, f_k]$ and $\langle G_2\rangle \cap \mathbb{Q}[f_0, f_1, \ldots, f_k]$ will have radical ideals differing for all values of $k > M + N + 1$, seeing as G_2 vanishes everywhere on the variety $f_{M+1} = \cdots = f_{k-1} = 0$ but G_1 does not.

For use in the next subsection, we give the following auxiliary fact.

Proposition 4.3. *Suppose that R is a unique factorization domain and $\mathsf{ff}(R)$ is its fraction field. Let $b_1, b_2, \ldots, b_n \in \mathsf{ff}(R)$. Suppose that $f_0, f_1, f_2, \ldots$ all reside in R and satisfy the recurrence relationship*

$$f_k = b_1 f_{k-1} + b_2 f_{k-2} + \cdots + b_n f_{k-n} \tag{32}$$

for $k \geq n$. Also assume that

$$\begin{vmatrix} f_{n-1} & f_{n-2} & \cdots & f_0 \\ f_n & f_{n-1} & \cdots & f_1 \\ \vdots & \vdots & \ddots & \vdots \\ f_{2n-2} & f_{2n-3} & \cdots & f_{n-1} \end{vmatrix} \neq 0. \tag{33}$$

Then $b_1, b_2, \ldots, b_n \in R$.

Proof. For $k_1, k_2, \ldots, k_n \geq n-1$, define

$$\Delta_{k_1,k_2,\ldots,k_n} = \begin{vmatrix} f_{k_1} & f_{k_1-1} & \cdots & f_{k_1-n+1} \\ f_{k_2} & f_{k_2-1} & \cdots & f_{k_2-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k_n} & f_{k_n-1} & \cdots & f_{k_n-n+1} \end{vmatrix}, \tag{34}$$

which belongs to R. For terms in the formal polynomial ring $\mathbb{Z}[b_1, b_2, \ldots, b_n]$ we consider the monomial ordering given by graded reverse lexicographical order using the precedence $b_1 < b_2 < \cdots < b_n$ [2].

Claim: For $k_n > k_{n-1} > \cdots > k_1 \geq n-1$, there exists a polynomial $B_{k_1,k_2,\ldots,k_n}$ in $\mathbb{Z}[b_1, b_2, \ldots, b_n]$ with leading term $\pm b_1^{k_n-k_{n-1}-1} b_2^{k_{n-1}-k_{n-2}-1} \cdots b_{n-1}^{k_2-k_1-1} b_n^{k_1-n+1}$

such that

$$\Delta_{k_1,k_2,\ldots,k_n} = B_{k_1,k_2,\ldots,k_n}\Delta_{n-1,n,\ldots,2n-2}. \tag{35}$$

whenever the recurrence relation (32) holds.

Once the claim is established, it follows by a process of backwards substitution that $b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n} \Delta_{n-1,n,\ldots,2n-2}$ (for $\ell_1, \ell_2, \ldots \ell_n \geq 0$) may be expressed as a $\mathbb{Z}$ linear combination of various $\Delta_{k_1,k_2,\ldots,k_n}$, all of which reside in R. So suppose, for sake of contradiction, that some b_j does not belong to R but rather equals $\frac{p}{q}$ for $p, q \in R$ where some irreducible $r \in R$ divides q but not p. Owing to (33), there is some finite value ℓ such that r^ℓ divides $\Delta_{n-1,n,\ldots,2n-2}$ but $r^{\ell+1}$ does not. However $b_j^{\ell+1}\Delta_{n-1,n,\ldots,2n-2}$ being in R implies that $r^{\ell+1}$ divides $p^{\ell+1}\Delta_{n-1,n,\ldots,2n-2}$, which yields the desired contradiction.

So it suffices to prove the claim, which we do via induction on k_n. When $k_n = 2n-2$, it follows that each k_j must equal $n-2+j$. So $\Delta_{k_1,k_2,\ldots,k_n} = \Delta_{n-1,n,\ldots,2n-2}$, thus we simply set $B_{k_1,k_2,\ldots,k_n} = 1$.

Let $k_n > 2n-2$ and assume that the claim holds for lesser values of k_n. By the recurrence relation and linearity of (34) with respect to its last row, it holds that

$$\Delta_{k_1,\ldots,k_{n-1},k_n} = b_1\Delta_{k_1,\ldots,k_{n-1},k_n-1} + b_2\Delta_{k_1,\ldots,k_{n-1},k_n-2} + \cdots + b_n\Delta_{k_1\ldots,k_{n-1},k_n-n}. \tag{36}$$

Each term on the right-hand side of this equation is zero or falls under the scope of the inductive hypothesis after appropriate row swaps. So we can recursively define the needed $B_{k_1,k_2,\ldots,k_n} \in \mathbb{Z}[b_1, b_2, \ldots, b_n]$ by setting

$$B_{k_1,\ldots,k_{n-1},k_n} = b_1 B_{k_1,\ldots,k_{n-1},k_n-1} + b_2 B_{k_1,\ldots,k_{n-1},k_n-2} + \cdots + b_n B_{k_1,\ldots,k_{n-1},k_n-n}, \tag{37}$$

where it is understood that permuting indices in the subscript of $B_{k_1,k_2,\ldots,k_n}$ changes its sign according to the sign of the permutation and that repeated indices causes $B_{k_1,k_2,\ldots,k_n}$ to be zero. It only remains to verify the claimed leading term of (37).

Let p denote the largest number such that $k_{p-1} < k_p - 1$, or let $p = 1$ if $k_j = k_n - n + j$ for all j, in which case $k_1 > n-1$. First consider the case $p = n$. When $B_{k_1,\ldots,k_{n-1},k_n-j}$ is non-zero, it follows from the inductive hypothesis that the degree of its leading term is $\max(k_{n-1}, k_n - j) - 2n + 2$. Thus $b_1 B_{k_1,\ldots,k_{n-1},k_n-1}$ (which is non-zero) has the greatest leading term, namely $\pm b_1^{k_n-k_{n-1}-1} \cdots b_{n-1}^{k_2-k_1-1} b_n^{k_1-n+1}$, owing to degree.

Now consider the case $p < n$. Notice that $B_{k_1,\ldots,k_{n-1},k_n-j}$ is zero for $j \leq n-p$. Also the leading terms of each $b_j B_{k_1,\ldots,k_{n-1},k_n-j}$ (when non-zero) share the same degree, namely $k_{n-1} - 2n + 3$, and they all omit b_ℓ for all $\ell < n-p$. But for $j \geq n-p+2$, the leading terms of each nonzero $b_j B_{k_1,\ldots,k_{n-1},k_n-j}$ are divisible by b_{n-p}. But the leading term of $b_{n-p+1} B_{k_1,\ldots,k_{n-1},k_n-n+p-1}$ is $\pm b_{n-p+1}^{k_p-k_{p-1}-1} \cdots b_{n-1}^{k_2-k_1-1} b_n^{k_1-n+1}$, which does not contain b_{n-p}. Thus this is the leading term of (37), due to the given monomial ordering.

$\square$

Remark: If $\Delta_{n-1,n,\ldots,2n-2} = 0$, then it follows from the claim in the proof above that $\Delta_{k_1,k_2,\ldots,k_n} = 0$ for all $k_n > k_{n-1} > \cdots > k_1 \geq n-1$. This implies that there exist $b'_1, b'_2, \ldots, b'_n \in \mathsf{ff}(R)$, not all zero, such that

$$b'_1 f_{k-1} + b'_2 f_{k-2} + \cdots + b'_n f_{k-n} = 0 \tag{38}$$

for $k \geq n$. Thus one may reduce to the case of recurrence relationship of lesser order, and one may repeat this reduction until (33) holds (so long as some f_k is non-zero). But, while this proposition holds for the coefficients in the reduced recurrence relation, it does not imply that the result holds for the coefficients of the original recurrence relation. For example, consider

$$f_k = (x - \frac{1}{x}) f_{k-1} + f_{k-2} \tag{39}$$

which is satisfied by $f_k = x^k f_0$. But this can also be defined using the lower order recurrence relation $f_k = x f_{k-1}$.

4.2. **Rationality Criteria on Taylor Series Using Radial Slices.** To consider radial slices algebraically, we define the map $\eta : \mathcal{C}[[\xi_1, \xi_2, \ldots, \xi_m]] \to \mathcal{C}[[t, \lambda_1, \lambda_2, \ldots, \lambda_m]]$ by saying $\eta(\sum_\alpha f_\alpha \xi^\alpha) = \sum_\alpha f_\alpha (t\lambda)^\alpha = \sum_\alpha f_\alpha t^{|\alpha|} \lambda^\alpha$. Since η is injective, it has a natural extension to the fraction field $\mathcal{C}((\xi_1, \xi_2, \ldots, \xi_m))$. For a fixed $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*) \in \mathcal{C}^m$, we define the substitution homomorphism $\pi_{\lambda^*} : \mathcal{C}[[t, \lambda_1, \lambda_2, \ldots, \lambda_m]] \to \mathcal{C}[[t]]$ by saying $\pi_{\lambda^*}(\sum_{j,\alpha} f_{j,\alpha} t^j \lambda^\alpha) = \sum_j (\sum_\alpha f_{j,\alpha} (\lambda^*)^\alpha) t^j$. (Note π_{λ^*} can be extended to the localization $(\pi_{\lambda^*}^{-1}(\mathcal{C}[[t]] \backslash \{0\}))^{-1} \mathcal{C}[[t, \lambda_1, \lambda_2, \ldots, \lambda_m]]$, but not to $\mathcal{C}((t, \lambda_1, \lambda_2, \ldots, \lambda_m))$ seeing as some outputs would have to be considered infinite or indeterminate.) Geometrically speaking, $\pi_{\lambda^*}(\eta(F))$ corresponds to the radial slice of the function F along the line $\xi = t\lambda^*$ (assuming $\lambda^* \neq 0$), while $\eta(F)$ captures the slicing data for all λ^*.

Define

$$G_j = \sum_{|\alpha|=j} \frac{1}{\alpha!} \lambda^\alpha E(D^\alpha F) \tag{40}$$

for $j \geq 0$ and set $G_j = 0$ for $j < 0$. These are in fact the Taylor coefficients of $\eta(F)$ with respect to t, that is

$$\eta(F) = G_0 + G_1 t + G_2 t^2 + \cdots . \tag{41}$$

To see this, let E_t denote the evaluation on $\mathcal{C}((t, \lambda_1, \lambda_2, \ldots, \lambda_m))$ (considered as a differential ring with simply the derivation D_t) that substitutes 0 for t, and observe that

$$\frac{1}{j!}E_t(D_t^j(\eta(F)) = \frac{1}{j!}E_t\left(\sum_{|\alpha|=j}\frac{j!}{\alpha!}\lambda^\alpha\eta(D^\alpha F)\right) = \sum_{|\alpha|=j}\frac{1}{\alpha!}\lambda^\alpha E(D^\alpha F) = G_j. \tag{42}$$

So we may present our third approach, which geometrically corresponds to characterizing rationality using rationality along radial one-dimensional slices.

Theorem 4.4. *Assume that* $\operatorname{char}\mathcal{M} = 0$ *and* $\mathcal{M}$ *has an evaluation with Taylor-regular domain* Dom_E*. Assume* $F \in \mathrm{Dom}_E$*. Let* $M, N \in \mathbb{N}$*. The following are equivalent*

1. *F is rational with respect to $\xi_1, \xi_2, \ldots, \xi_m$ with numerator and denominator having degrees at most M and N, respectively.*
2. *$\eta(F)$ is rational with respect to t with numerator and denominator having degrees at most M and N, respectively.*
3. *$\pi_{\lambda^*}(\eta(F))$ is rational with respect to t with numerator and denominator having degrees at most M and N, respectively, for all $\lambda^* \in \mathbb{C}^m$*
4. *For $M < k_1 < k_2 < \cdots < k_{N+1}$,*

$$E_t(\det \tilde{T}^{\{0,1,\ldots,N\}}_{\{k_1,k_2,\ldots,k_{N+1}\}}[\eta(F)]) = \det\begin{bmatrix} G_{k_1} & G_{k_1-1} & \cdots & G_{k_1-N} \\ G_{k_2} & G_{k_2-1} & \cdots & G_{k_2-N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{k_{N+1}} & G_{k_{N+1}-1} & \cdots & G_{k_{N+1}-N}\end{bmatrix} = 0 \tag{43}$$

5. *For $k \geq M$,*

$$\mathcal{H}_{k,N}[\eta(F)] = \det\begin{bmatrix} G_{k+1} & G_k & \cdots & G_{k-N+1} \\ G_{k+2} & G_{k+1} & \cdots & G_{k-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ G_{k+N+1} & G_{k+N} & \cdots & G_{k+1}\end{bmatrix} = 0. \tag{44}$$

Proof. ($1 \implies 2 \implies 3$) Clear.

($3 \implies 4 \iff 5 \iff 2$) Using Theorem 4.2, condition 3 implies that (43) holds fiberwise for all choices of $\lambda^* \in \mathbb{C}^M$. Thus (43) also holds in $\mathbb{C}[[\lambda_1, \lambda_2, \ldots, \lambda_m]]$. Moreover, conditions 5, 4, and 2 are equivalent in light of Theorem 4.2.

($5 \implies 1$) It suffices to consider the case where N is the minimal value for which condition 5 holds. As a consequence of Sylvester's identity (28) it follows that $\mathcal{H}_{k,N-1} \neq 0$ for $k \geq M-1$

Then using the formulas (29) and (30) we can calculate $a_0, a_1, \ldots, a_M$ and $b_0, b_1, \ldots, b_N$ in $\mathbb{C}[\lambda_1, \lambda_2, \ldots, \lambda_m]$ such that $\eta(F) = \frac{a_0+a_1t+\cdots+a_Mt^M}{b_0+b_1t+\cdots+b_Nt^N}$. This gives each a_j and b_j as homogeneous polynomials of degree $MN+j$, with $b_0 = \mathcal{H}_{M-1,N-1}[\eta(F)] \neq 0$.

Taking the t^k terms (for $k \geq \max(M+1, N)$) of the equation

$$(a_0 + a_1 t + \cdots + a_M t^M) = \eta(F)(b_0 + b_1 t + \cdots + b_N t^N), \tag{45}$$

allows one to produce the recurrence relation

$$G_k = -\frac{b_1}{b_0} G_{k-1} - \frac{b_2}{b_0} G_{k-2} - \cdots - \frac{b_N}{b_0} G_{k-N} \tag{46}$$

for $k \geq \max(M+1, N)$. Since all G_k reside in $\mathcal{C}[\lambda_1, \lambda_2, \ldots, \lambda_m]$, Proposition 4.3 implies that b_0 divides each b_j within $\mathcal{C}[\lambda_1, \lambda_2, \ldots, \lambda_m]$. And it follows from (30) that b_0 divides each a_j. Thus $\frac{b_j}{b_0}$ and $\frac{a_j}{b_0}$ are homogeneous polynomials of degree j in $\mathcal{C}[\lambda_1, \lambda_2, \ldots, \lambda_m]$. So there exist homogeneous p_j, q_j of degree j in $\mathcal{C}[\xi_1, \xi_2, \ldots, \xi_m]$ such that $\eta(q_j) = \frac{b_j}{b_0} t^j$ and $\eta(p_j) = \frac{a_j}{a_0} t^j$. Setting $P = \sum_{j=0}^{M} p_j$ and $Q = \sum_{j=0}^{N} q_j$, we obtain that $\eta(F) = \frac{\eta(P)}{\eta(Q)}$. Therefore $F = \frac{P}{Q}$, which satisfies condition 1.

□

Remarks:

1) The assumption $F \in \mathrm{Dom}_E$ makes conditions 3, 4 and 5 well-defined. But more critically, the equivalence of the degree bounds in conditions 1 and 2 can break down without this assumption. To demonstrate this, consider the example $F = \xi_1/\xi_2$, which has numerator and denominator both of degree 1. But $\eta(F) = \lambda_1/\lambda_2$ is constant with respect to t and so it is rational with numerator and denominator having degree 0 with respect to t. However the equivalence of conditions 1 and 2 does hold if F or $\frac{1}{F}$ belong to Dom_E.

2) We may extend a remark following Theorem 4.2 to the multivariable case. Namely when F satisfies any of these equivalent rationality conditions, then the Taylor coefficients of F of order $M+N+1$ or greater are uniquely determined by the Taylor coefficients with order between $M-N+1$ at most $M+N$, and the Taylor coefficients of order $M-N$ or less are arbitrary.

3) By expanding the relations (43) or (44) and taking coefficients with respect to the monomials of $\lambda_1, \lambda_2, \ldots, \lambda_m$, one generates a set of relations in $\mathbb{Q}[\{f_\alpha\}_{\alpha \in \mathbb{N}^m}]$. Each of these relations have degree $N+1$ in terms of $\{f_\alpha\}_{\alpha \in \mathbb{N}^m}$. Plus the coefficients of λ_j^p produce the rationality conditions purely with respect to ξ_j. So the relations produced here are in a sense "tighter" than the relations occurring in Theorem 4.1 and not subject to some of the reductions discussed following Theorem 4.1.

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Dept. of Computer Science and Mathematical Sciences, Penn State Harrisburg
777 West Harrisburg Pike, Middletown, PA 17057
E-mail: rawalker@psu.edu