BOUNDARIES OF HOLOMORPHIC 1-CHAINS WITHIN HOLOMORPHIC LINE BUNDLES OVER $\mathbb{CP}^1$

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Abstract. We show that boundaries of holomorphic 1-chains within holomorphic line bundles of $\mathbb{CP}^1$ can be characterized using a single generating function of Wermer moments.

In the case of negative line bundles, a rationality condition on the generating function plus the vanishing moment condition together form an equivalent condition for bounding. We provide some examples which reveal that the vanishing moment condition is not sufficient by itself. These examples also can be used to demonstrate one point of caution about the use of birational maps in this topic.

In the case of positive line bundles, where the vanishing moment condition vacuously holds, boundaries of holomorphic 1-chains can be characterized using the aforementioned rationality condition modulo a series of polynomial terms whose degrees are dependent on the degree of the line bundle.

As a side point with potential independent interest, we show for any meromorphic function that rationality with prescribed bounds on degree is equivalent to the satisfaction of a particular determinantal differential equation.

1. Introduction

One biholomorphic invariant of a complex space is its collection of boundaries of holomorphic p-chains. A well-known result by Harvey and Lawson [7] characterizes the real $(2p-1)$-chains, with certain regularity, that are boundaries of holomorphic p-chains within a Stein space by the vanishing moment condition or, if $p > 1$, by maximal complexity. (For reasons epitomized by this result, the study of boundaries of holomorphic p-chains often splits into the cases of $p > 1$ and $p = 1$. We subsequently focus on the case $p = 1$.) The collection of boundaries of holomorphic 1-chains within projective space differs fundamentally in character from the corresponding collection within affine space. (For instance, in $\mathbb{CP}^n$ any closed curve can be arbitrarily approximated by a closed curve that bounds a holomorphic

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1-chain within $\mathbb{CP}^n$.) Hence it is not surprising that characterizations of boundaries of holomorphic 1-chains within projective space, such as those in [2], [8], and [9], bear a different flavor from the affine case. At the same time, the referenced characterizations each feature a type of "correspondence principle", i.e. a way of extracting the usual affine characterizations by using suitable restrictions. In light of the contrast between affine and projective cases, we are interested in examining spaces intermediate to affine and projective space in a comparative fashion.

The purpose of this paper is develop characterizations of boundaries of holomorphic 1-chains within holomorphic line bundles over $\mathbb{CP}^1$, treated as spaces in their own right. This family of line bundles provides a spectrum of spaces intermediate to the affine and projective cases. Moreover, we harness a Wermer moment generating function used in [9], which provides a common vehicle for expressing characterizations across different line bundles in this class. A definite distinction between negative and positive line bundles emerges.

The negative line bundles over $\mathbb{CP}^1$ possess a non-trivial vanishing moment condition, which is necessary for bounding by Stokes' Theorem. But, as our examples will show, the vanishing moment condition is not sufficient in this case, thus differing from the case of a Stein space. (Since $\mathcal{O}(-1)$ and $\mathbb{C}^2$ are birationally equivalent via a basic blowdown map, our observations and examples also affirm a proviso regarding the use of birational maps in this topic.) Sufficiency may be gained by adding a rationality condition on our Wermer moment generating function or by requiring that the support of the real 1-chain avoid the zero section. We also note that this rationality condition can be expressed in terms of differential equations. (For comparison, a condition in terms of differential equations has also been developed, via very different means, that provides a characterization of boundaries of holomorphic 1-chains within $\mathbb{CP}^2$ in [10].)

For positive line bundles over $\mathbb{CP}^1$, boundaries of holomorphic 1-chains can be characterized by using the mentioned rationality criterion modulo certain polynomial terms dependent on the degree of the line bundle. (The vanishing moment condition is vacuous in this case.) As $\mathcal{O}(1)$ and $\mathbb{CP}^2\backslash(0:0:1)$ are biholomorphic, this also provides an extension of a special result occurring in [8].

We begin with preliminaries, including a discussion of some rationality criteria, in Section 2. We proceed to our main characterization results in Section 3, addressing the negative line bundle case in Subsection 3.1 and the positive line bundle case in Subsection 3.2.

2. Preliminaries

2.1. **Boundaries of Holomorphic 1-Chains.** Much of the standard definitions and the background of geometric measure theory can be found in a text by Federer

[3], or in an article by Harvey [6]; the latter is more tailored to the specific topics of this article.

Let X be a complex space with some given Hermitian metric. A holomorphic 1-chain V in X is a locally finite $\mathbb{Z}$-linear combination of currents of integration of analytic varieties in X with pure complex dimension one. (Alternatively, a holomorphic 1-chain V can be viewed as an analytic variety of pure complex dimension one, i.e. its support $\operatorname{spt} V$, with integer multiplicities associated to each component. Then one tacitly identifies this formal object with its corresponding current of integration.)

For a closed, rectifiable 1-current γ and a holomorphic 1-chain V in $X \backslash \operatorname{spt} \gamma$ such that V has a simple (or trivial) extension as a current to X, we say that γ is the *boundary of V within X* if

- $dV = \gamma$ (as currents in X), and
- $\operatorname{spt} V \Subset X$.

Remark: For V to have a simple extension it is sufficient that V have finite mass. Conversely, if V has a simple extension and it is bounded by γ, then V has finite mass. So the condition that V have finite mass is a suitable surrogate for the requirement that V have a simple extension to X. Also note that the statement "γ bounds V within X" is independent of the choice of Hermitian metric on X.

Let $\phi : Z \to Y$ be holomorphic map between two connected complex spaces. The map ϕ is called a *proper modification* if ϕ is proper and there exists a dense Zariski open set U_Y in Y such that $U_Z = \phi^{-1}(U_Y)$ is dense in Z and $\phi|_{U_Z}$ is a biholomorphism. U_Z can be chosen to be maximal, in which case $Z \backslash U_Z$ and $Y \backslash U_Y$ are called, respectively, the exceptional set and indeterminacy set (or center) of the modification. (We refer to [4] and [5] for further background.)

A *bimeromorphic map* $f : X \dashrightarrow Y$ between X and Y can be specified by giving its graph Γ_f as an analytic subspace of $X \times Y$ such that the natural projections $\pi_1 : \Gamma_f \to X$ and $\pi_2 : \Gamma_f \to Y$ are proper modifications. Then f is in spirit identified with $\pi_2 \circ \pi_1^{-1}$. While this is not a true function at all points in X, it does yield a biholomorphic function at generic points in X. To state this more precisely, we introduce the following definitions. Let the *indeterminacy set of f*, denoted $\mathcal{I}(f)$, be the indeterminacy set of the proper modification π_1, i.e. those points in X where π_1 does not possess a local inverse. (If we view π_1 as bimeromorphic function, then $\mathcal{I}(f) = \mathcal{I}(\pi_1^{-1})$. This reveals a conceptual reversal in the technical definition of indeterminacy sets for proper modifications. For the sake of clarity, from this point forward we shall impose the context of bimeromorphic maps when considering indeterminacy sets.) We define the *exceptional set of f by* $\mathcal{E}(f)$ to be the π_1-projection of the exceptional set of π_2, i.e. $\mathcal{E}(f) = \pi_1(\mathcal{E}(\pi_2)) = \pi_1(\pi_2^{-1}(\mathcal{I}(\pi_2^{-1})))$.

Then $f = \pi_2 \circ \pi_1^{-1}$, suitably restricted, defines a biholomorphic function between the two dense Zariski open subsets $U_X = X \backslash (\mathcal{I}(f) \cup \mathcal{E}(f))$ and $U_Y = Y \backslash (\mathcal{I}(f^{-1}) \cup \mathcal{E}(f^{-1}))$ of X and Y, respectively.

Proposition 2.1. *Let $f : X \dashrightarrow Y$ be a bimeromorphic map. If γ is a rectifiable 1-current in X such that $\operatorname{spt}\gamma \Subset U_X = X \backslash (\mathcal{E}(f) \cup \mathcal{I}(f))$. Then γ bounds a holomorphic 1-chain within X if and only if $f_*(\gamma)$ bounds a holomorphic 1-chain within Y.*

As examples in Subsection 3.1 will show, the assumption that $\operatorname{spt}\gamma$ avoid the exceptional and indeterminacy sets of f cannot be removed in general. Also we present this proof in a way that is readily generalizable to boundaries of holomorphic p-chains.

Proof. It suffices to consider the case where f is a proper modification. Note that by the support assumptions on γ it follows that $f|_{X \backslash \operatorname{spt}\gamma}$ is a proper modification onto $Y \backslash f(\operatorname{spt}\gamma)$. By Remmert's proper mapping theorem, for an irreducible analytic set A in X (resp. $X \backslash \operatorname{spt}\gamma$), its image $f(A)$ is an analytic set in Y (resp. $Y \backslash f(\operatorname{spt}\gamma)$). The set $\overline{f|_{U_X}(A \cap U_X)}$, which is $f(A)$ minus any components contained wholly in $Y \backslash U_Y$, has the same dimension as A. Thus given a holomorphic 1-chain V in $X \backslash \operatorname{spt}\gamma$ that is compactly supported in X, it holds that $V' = \overline{(f|_{U_X \backslash \operatorname{spt}\gamma})_*(V \cap (U_X \backslash \operatorname{spt}\gamma))}$ defines a holomorphic 1-chain in $Y \backslash f(\operatorname{spt}\gamma)$ that is compactly supported in Y. V' has no boundary in $Y \backslash f(\operatorname{spt}\gamma)$, but by consideration of the biholomorphism $f|_{U_X}$, if $dV = \gamma$ then $dV' = f_*(\gamma)$, when V' is viewed as a current in Y.

For the reverse, we see that an irreducible analytic set A in Y naturally defines an analytic set in X via the pull-back $f^*(A)$. The proper transform $\overline{f^{-1}(A \cap U_Y)}$ also defines an analytic set in X, but the removal of any components contained in $X \backslash U_X$ ensures that the proper transform has the same dimension as A. By extension, the proper transform of a holomorphic 1-chain bounded by $f_*(\gamma)$ within Y forms a holomorphic 1-chain bounded by γ within X.

□

2.2. **Algebraic preliminaries.** For $(\zeta^*, \xi^*) \in \mathbb{CP}^1 \times \mathbb{CP}^1$, let $\mathcal{O}_{(\zeta^*,\xi^*)}$ denote the ring of germs of holomorphic functions about (ζ^*, ξ^*), and let $\mathcal{M}_{(\zeta^*,\xi^*)}$ denote the associated fraction field, i.e. the field of germs of meromorphic functions. Let $\mathcal{O}^*_{(\zeta^*,\xi^*)}$ and $\mathcal{M}^*_{(\zeta^*,\xi^*)}$ denote the group of germs of nonvanishing holomorphic functions and the group of germs of non-identically zero meromorphic functions, respectively, about (ζ^*, ξ^*). Similarly define the one-dimensional analogs $\mathcal{O}_{\zeta^*}$, $\mathcal{O}_{\xi^*}$, $\mathcal{M}_{\zeta^*}$, etc. The field of germs of rational functions with respect to ξ about ζ^*, denoted $\mathcal{O}_{\zeta^*}(\xi)$, is the fraction field of the polynomial ring $\mathcal{O}_{\zeta^*}[\xi]$. It has a canonical inclusion into $\mathcal{M}_{(\zeta^*,\xi^*)}$ for any ξ^* in $\mathbb{CP}^1$.

For $S \in \mathcal{O}_{\zeta^*}(\xi)$, the divisor of S yields a germ of a holomorphic 1-chain V near $z = \zeta^*$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Conversely for any germ of a holomorphic 1-chain V near $z = \zeta^*$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$, there exists a $S \in \mathcal{O}_{\zeta^*}(\xi)$ whose divisor is $V + kO$ for some integer k, where O is the germ of the variety $\xi = 0$. (In particular k will have the value so that the total degree of intersection of $V + kO$ with vertical lines is zero.)

Note that restriction to $\xi = \xi^*$ yields a well-defined map from $\mathcal{M}_{(\zeta^*,\xi^*)} \cup \{\infty\}$ to $\mathcal{M}_{\zeta^*} \cup \{\infty\}$. We say that $S \in \mathcal{M}^*_{(\zeta^*,\infty)}$ is *in normalized form* if $S|_{\xi=\infty} \in \mathcal{M}^*_{\zeta^*}$. Additionally, we say that S is in (N1) form if $S|_{\xi=\infty} = 1$ and we say that S is in (N2) form if $S|_{\xi=\infty} \in \mathcal{M}^*_{\zeta^*}$ and $S|_{\zeta=\zeta^*} \in \mathcal{M}^*_{\infty}$. (Geometrically, S is in (N2) form if and only if the divisor of S has no components in $\xi = \infty$ or $\zeta = \zeta^*$.) For $S \in \mathcal{O}_{\zeta^*}(\xi)$, S can be converted into normalized form by multiplication by a suitable power of ξ, and it can be placed into (N2) form by further multiplication by a suitable power of $(\zeta - \zeta^*)$ (or a suitable power of $\frac{1}{\zeta}$ if $\zeta^* = \infty$). These normalization operations are closed on both $\mathcal{O}_{\zeta^*}(\xi)$ and $\mathcal{O}_\infty(\zeta)$. (For (N1) form instead, one generally requires multiplication by an element of $\mathcal{M}^*_{\zeta^*}$.)

The ξ-logarithmic derivative of S arises in the Newton formula of symmetric function theory. In particular, if $S = 1 - e_1\xi^{-1} + \cdots + (-1)^n e_n \xi^{-n}$ is a generating function of the elementary symmetric functions for a given finite multiset, then $\frac{S_\xi}{S}$ equals $\sum_{k=1}^\infty c_k \xi^{-k-1}$, a generating function of the sums of powers of the multiset. (This is applicable to a finite multiset in any ring or field, but we focus on fields of the form $\mathcal{M}_{\zeta^*}$ at the present.) There is a straightforward algebraic extension of this to the case where $S \in \mathbb{C}(\xi)$ and S is in (N1) form, wherein one considers finite 0-chains rather than finite multisets.

At times we desire to convert information about a function to information about its logarithmic derivative and vice versa. In that light, we state the following proposition whose proof is basic.

Proposition 2.2. *Let $f \in \mathcal{M}^*_{(\zeta^*,\xi^*)}$. If $\xi^* \neq \infty$, then $f \in \mathcal{O}^*_{\zeta^*,\xi^*}$ if and only if $\frac{f_\xi}{f} \in \mathcal{O}_{(\zeta^*,\xi^*)}$ and $f|_{\xi=\xi^*} \in \mathcal{O}^*_{\zeta^*}$. In the case that $\xi^* = \infty$, then $f \in \mathcal{O}^*_{\zeta^*,\infty}$ if and only if $\frac{f_\xi}{f} \in \frac{1}{\xi^2}\mathcal{O}_{(\zeta^*,\infty)}$ and $f|_{\xi=\infty} \in \mathcal{O}^*_{\zeta^*}$.*

2.3. **Differential Criteria for Rationality.** For $F \in \mathcal{M}_{(\zeta^*,\xi^*)}$, define

$$\mathcal{S}^F_{M,N} = \det \begin{bmatrix} \frac{D_\xi^{M+1}F}{(M+1)!} & \frac{D_\xi^{M}F}{M!} & \cdots & \frac{D_\xi^{M-N+1}F}{(M-N+1)!} \\ \frac{D_\xi^{M+2}F}{(M+2)!} & \frac{D_\xi^{M+1}F}{(M+1)!} & \cdots & \frac{D_\xi^{M-N+2}F}{(M-N+2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{D_\xi^{M+N+1}F}{(M+N+1)!} & \frac{D_\xi^{M+N}F}{(M+N)!} & \cdots & \frac{D_\xi^{M+1}F}{(M+1)!} \end{bmatrix}, \tag{1}$$

replacing any entry $\frac{D_\xi^q F}{q!}$ with zero for $q < 0$. (For the purposes of later identities, define $\mathcal{S}^F_{M,-1} = 1$.) This expression is central to the following differential characterization of rational functions with prescribed bounds on the degrees of the numerator and denominator.

Proposition 2.3. *F is a rational function of ξ with numerator having degree at most M and denominator having degree at most N if and only if $\mathcal{S}^F_{M,N} = 0$.*

Remark: The statement $\mathcal{S}^F_{1,1} = 0$ is equivalent to saying that the Schwarzian derivative of F is zero. So Proposition 2.3 generalizes the Schwarzian derivative characterization of linear fractional functions.

Proof. F is a rational function with the mentioned restrictions on degree if and only if the functions $1, \xi, \ldots \xi^M, F, \xi F, \ldots \xi^N F$ are linearly dependent over $\mathcal{M}_{\zeta^*}$. The latter is equivalent to the identical vanishing of the Wronskian, considered with respect to ξ, for the indicated set of functions. This Wronskian equals $\prod_{p=0}^{M} p!$ times the determinant

$$\det \begin{bmatrix} D_\xi^{M+1}F & D_\xi^{M+1}(\xi F) & \cdots & D_\xi^{M+1}(\xi^N F) \\ D_\xi^{M+2}F & D_\xi^{M+2}(\xi F) & \cdots & D_\xi^{M+2}(\xi^N F) \\ \vdots & \vdots & \ddots & \vdots \\ D_\xi^{M+N+1}F & D_\xi^{M+N+1}(\xi F) & \cdots & D_\xi^{M+N+1}(\xi^N F) \end{bmatrix} \tag{2}$$

By application of the product rule and suitable column operations, followed by appropriate scalar multiplication of the rows, it can be shown that the determinant in (2) equals $\prod_{p=M+1}^{M+N+1} p!$ times $\mathcal{S}^F_{M,N}$. ☐

This condition can also be phrased as a differential condition in terms of the logarithmic derivative $H = \frac{F_\xi}{F}$. For $H \in \frac{1}{\xi^2}\mathcal{O}_{(\zeta_0,\infty)}$, let

$$\mathcal{T}^H_{M,N} = \det \begin{bmatrix} \frac{T_{M+1}}{(M+1)!} & \frac{T_M}{M!} & \cdots & \frac{T_{M-N+1}}{(M-N+1)!} \\ \frac{T_{M+2}}{(M+2)!} & \frac{T_{M+1}}{(M+1)!} & \cdots & \frac{T_{M-N+2}}{(M-N+2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{T_{M+N+1}}{(M+N+1)!} & \frac{T_{M+N}}{(M+N)!} & \cdots & \frac{T_{M+1}}{(M+1)!} \end{bmatrix}, \tag{3}$$

where $T_j = 0$ for $j < 0$, $T_0 = 1$, $T_1 = -\xi^2 H$, and $T_j = \frac{\partial}{\partial \frac{1}{\xi}}(T_{j-1}) + T_1 T_{j-1}$ for $j > 1$.

Proposition 2.4. *Let $H \in \mathcal{M}_{(\zeta^*,\infty)}$ with $H \in \frac{1}{\xi^2}\mathcal{O}_{(\zeta_0,\infty)}$ for some ζ_0 near ζ^*. There exists a $S \in \mathcal{O}_{\zeta^*}(\xi) \cap \mathcal{M}^*_{(\zeta^*,\infty)}$ with numerator having degree at most M and denominator having degree at most N, with respect to $\frac{1}{\xi}$, such that $H = \frac{S_\xi}{S}$ if and only if $\mathcal{T}^H_{M,N} = 0$.*

Proof. Define $F(\zeta,\xi) = \exp\left[\int_{\xi^*}^{\xi} H(\zeta,\xi')\, d\xi'\right]$. Note that if $S \in \mathcal{O}_{\zeta^*}(\xi) \cap \mathcal{M}^*_{(\zeta^*,\infty)}$ and $\frac{S_\xi}{S} = H$, then $S = AF$, for some function $A \in \mathcal{M}^*_{\zeta^*}$. Apply Proposition 2.3 to F for rationality in terms of $\frac{1}{\xi}$. By dividing each of the entries in the determinant

$\mathcal{S}^F_{M,N}$ by F and by observing inductively that $T_j = \frac{D^j_{1/\xi}F}{F}$ in the definition of $\mathcal{T}^H_{M,N}$, the proposition follows.

□

Further observations:

(1) Define $\mathcal{H}^F_{M,N} = \mathcal{S}^F_{M,N}|_{\xi=\xi^*}$. For $\xi^* \neq \infty$, the classical Hadamard criterion for rationality with can be expressed as saying that $F \in \mathcal{O}_{\xi^*}$ is a rational function of ξ with numerator having degree at most M and denominator having degree at most N if and only if $\mathcal{H}_{M+j,N} = 0$ for all $j \geq 0$.

(2) Note that $\mathcal{S}^F_{M,N} = 0$ implies that $\mathcal{S}^F_{M+j,N+k} = 0$ for every $j, k \geq 0$, which is a simple result of Proposition 2.3. By Sylvester's determinant identity, it holds that $\left(\mathcal{S}^F_{M,N}\right)^2 - \mathcal{S}^F_{M-1,N}\mathcal{S}^F_{M+1,N} = \mathcal{S}^F_{M,N+1}\mathcal{S}^F_{M,N-1}$. So $\mathcal{S}^F_{M,N+1} = 0$ and $\mathcal{S}^F_{M+1,N} = 0$ together imply that $\mathcal{S}^F_{M,N} = 0$. From these observations, it follows that if $\mathcal{S}^F_{M,N}$ equals zero for some M and N, then there exists a unique pair $(\tilde{M}, \tilde{N})$ such that $\mathcal{S}^F_{M,N} = 0$ if and only if $M \geq \tilde{M}$ and $N \geq \tilde{N}$.

With $\tilde{M}$ and $\tilde{N}$ so defined, $\mathcal{H}^F_{M,\tilde{N}-1}$ is not identically zero for $M \geq \tilde{M}-1$. For, if $\mathcal{H}^F_{M,\tilde{N}-1}$ did equal 0, then Sylvester's identity could be successively applied to show that $\mathcal{H}^F_{M+j,\tilde{N}-1} = 0$ for $j \geq 0$. Employing the Hadamard criterion, it would follow that $\mathcal{S}^F_{M,\tilde{N}-1} = 0$, which yields a contradiction. It follows by a related, but simpler, argument that $\mathcal{H}^F_{\tilde{M}-1,N}$ is not identically zero for $N \geq \tilde{N}-1$, barring the case when F is identically zero and $\tilde{M} = 0$.

(3) Assume $\xi^* \neq \infty$ and let $F(\zeta, \xi) = \sum_{j=0}^{\infty} e_j(\zeta)(\xi - \xi^*)^j$ with $e_j(\zeta) \in \mathcal{M}_{\zeta^*}$, assuming uniform convergence in a neighborhood of (ζ_0, ξ^*), for some ζ_0 near ζ^*. It is straightforward to see that the coefficient of $(\xi - \xi^*)^j$ in the power series expansion of $\mathcal{S}^F_{M,N}$ has the form $\binom{M+N+1+j}{M+N+1} e_{M+N+1+j}\mathcal{H}^F_{M-1,N-1} + P_{M,N,j}$, where $P_{M,N,j}$ denotes some polynomial expression in terms of e_k for $M - N + 1 \leq k < M + N + 1 + j$. So, if $\mathcal{H}^F_{M-1,N-1}$ does not identically equal zero, then $\mathcal{S}^F_{M,N} = 0$ if and only if $e_{M+N+1+j} = \frac{-P_{M,N,j}}{\binom{M+N+1+j}{M+N+1}\mathcal{H}^F_{M-1,N-1}}$ for $j \geq 0$. Consequentially, if one arbitrarily chooses e_j for $j \leq M + N$ subject to the restraint that $\mathcal{H}^F_{M-1,N-1}$ not be identically zero, then the equation $\mathcal{S}^F_{M,N} = 0$ uniquely determines e_j for $j > M + N$.

(4) By incorporating the Newton formula, the previous observation can be used to show that if $H = \sum_{j=1}^{\infty} c_j \frac{1}{\xi^{j+1}}$ satisfies $\mathcal{T}^H_{M,N} = 0$ with a given M and N, then c_j for $j \geq M + N + 1$ is uniquely determined by c_j for $j \leq M + N$, so long as $\mathcal{T}^F_{M,N}|_{\xi=\infty} \neq 0$.

2.4. **Holomorphic line bundles.** Let $\mathcal{O}(n)$ denote the holomorphic line bundle of degree n over $\mathbb{CP}^1$. More explicitly, let $U_0 = \mathbb{C}$ and $U_\infty = \mathbb{CP}^1 \backslash \{0\}$ and define $\phi(z, w) = (z, z^{-n}w)$ on $(U_0 \cap U_\infty) \times \mathbb{C}$. As is known, $\mathcal{O}(n)$ can be defined by patching these two trivializations together, i.e. by identifying each point (z, w), for $z \neq 0$, in $U_0 \times \mathbb{C}$ with $\phi(z, w)$ in $U_\infty \times \mathbb{C}$. This identification is diagrammed below.

$$\begin{array}{ccc} U_0 \times \mathbb{C} & & U_\infty \times \mathbb{C} \\ (z,w) & \xrightarrow{\phi} & (z, z^{-n}w) \\ (z, z^n w) & \xleftarrow{\phi^{-1}} & (z,w) \end{array} \tag{4}$$

The choice of Hermitian metrics is largely immaterial to the results of this article, so the reader is free to select Hermitian metrics on $\mathcal{O}(n)$ as one desires. Determining the most "natural" metrics for these spaces may well be a topic for further consideration. To be concrete, we suggest the following metrics. For $\mathcal{O}(-n)$, $n > 0$, consider the Hermitian metric defined by the continuous extension of the closed (1,1) form $\frac{\mathrm{i}}{2\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2} + \frac{\mathrm{i}}{2}dw \wedge d\bar{w} + \frac{\mathrm{i}}{2}d(wz^n) \wedge d(\overline{wz^n})$ on $U_0 \times \mathbb{C}$. For the trivial bundle $\mathcal{O}(0)$, use the metric determined by the (1,1) form $\frac{\mathrm{i}}{2\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2} + \frac{\mathrm{i}}{2}dw \wedge d\bar{w}$. For $\mathcal{O}(n)$, $n > 0$, consider the Hermitian metric given by the (1,1) form $\frac{\mathrm{i}}{2\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2} + \frac{\mathrm{i}}{2}\frac{dw\wedge d\bar{w}}{(1+|z|^2)^{n+1}} + \frac{\mathrm{i}}{2}\frac{d(wz^{-n})\wedge d(\overline{wz^{-n}})}{(1+|z|^{-2})^{n+1}}$ on $U_0 \times \mathbb{C}$. These particular metrics are complete and symmetric under inversion between U_0 and U_∞. In the case of non-negative line bundles, the given metrics are Kähler.

Lemma 2.5. *Let V be a germ of a holomorphic 1-chain about $z = \infty$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ defined as the divisor of $S(z, w)$, where S is an element of $\mathcal{O}_\infty(\xi)$ in (N2) form. The following are equivalent:*

1. *$V \cap (U_0 \times \mathbb{C})$, as viewed in $U_0 \times \mathbb{C} \subseteq \mathcal{O}(n)$, extends to a germ of a holomorphic 1-chain about $z = \infty$ in $\mathcal{O}(n)$ with relatively compact support,*
2. *The proper transform of V under ϕ (viewed as a birational map from $\mathbb{CP}^1 \times \mathbb{CP}^1$ to itself) has relatively compact support in $K \times \mathbb{C}$, for some closed neighborhood K of ∞,*
3. *$\zeta^M S(\zeta, \xi\zeta^n)$ is an element of $\mathcal{O}^*_{(\infty,\infty)}$ for some integer M, and*
4. *$\zeta^n \frac{S_\xi}{S}(\zeta, \xi\zeta^n)$ is an element of $\frac{1}{\xi^2}\mathcal{O}_{(\infty,\infty)}$.*

Proof. By considering the definitions of $\mathcal{O}(n)$ and the proper transform of ϕ, the equivalence of 1 and 2 follows. The equivalence of 2 and 3 holds by considering the proper transform of ϕ algebraically. The equivalence of 3 and 4 is due to Proposition 2.2.

□

3. Boundaries of holomorphic 1-chains within holomorphic line bundles over $\mathbb{CP}^1$

Let γ be a closed, rectifiable 1-current in $\mathbb{CP}^1 \times \mathbb{CP}^1$ with support satisfying condition A_1. (For the definition of condition A_1, see [1].) If we assume that

$\operatorname{spt}\gamma \Subset \mathbb{C}^2$, then the integrals $\frac{1}{2\pi\mathrm{i}}\int_\gamma w^k z^j dz$ are well-defined for any non-negative integers j and k. We call these integrals the *Wermer moments* of γ in $\mathbb{C}^2$ [11]. Define $\hat{\mathfrak{U}}^\zeta_\gamma = \mathbb{CP}^1 \backslash \pi_1(\operatorname{spt}\gamma)$ and $\hat{\mathfrak{U}}^\xi_\gamma = \mathbb{CP}^1 \backslash \pi_2(\operatorname{spt}\gamma)$. For $(\zeta,\xi) \in \hat{\mathfrak{U}}^\zeta_\gamma \times \hat{\mathfrak{U}}^\xi_\gamma$, let

$$H_\gamma(\zeta,\xi) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{1}{\xi - w}\frac{dz}{z-\zeta}, \tag{5}$$

which is defined if $\operatorname{spt}\gamma \Subset \mathbb{C}\times\mathbb{CP}^1$, and let

$$G_\gamma(\zeta,\xi) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{1}{\xi - w}\frac{dz}{(z-\zeta)^2}. \tag{6}$$

(Technically speaking, for a meromorphic form ω, the integral $\int_\gamma \omega$ means the current γ evaluated against the form $\chi\omega$, where χ is some compactly supported C^∞ cut-off function that equals one on a neighborhood of $\operatorname{spt}\gamma$ and zero near any poles of ω.) These define holomorphic functions for (ζ,ξ) on $\hat{\mathfrak{U}}^\zeta_\gamma \times \hat{\mathfrak{U}}^\xi_\gamma$. Observe that $H_\gamma(\zeta,\xi)$ has the series expansion

$$H_\gamma(\zeta,\xi) = \sum_{k=0}^\infty\sum_{j=0}^\infty \left(\frac{-1}{2\pi\mathrm{i}}\int_\gamma w^k z^j dz\right)\frac{1}{\zeta^{j+1}}\frac{1}{\xi^{k+1}} \tag{7}$$

for (ζ,ξ) near (∞,∞) if $\operatorname{spt}\gamma \Subset \mathbb{C}^2$ and the series expansion

$$H_\gamma(\zeta,\xi) = \sum_{k=0}^\infty\sum_{j=0}^\infty \left(\frac{-1}{2\pi\mathrm{i}}\int_\gamma w^k \frac{1}{(z-\zeta^*)^{j-1}} d(\frac{1}{z-\zeta^*})\right)(\zeta-\zeta^*)^j\frac{1}{\xi^{k+1}} \tag{8}$$

for (ζ,ξ) near (ζ^*,∞) for $\zeta^* \in \hat{\mathfrak{U}}^\zeta_\gamma$ if $\operatorname{spt}\gamma \Subset \mathbb{CP}^1\backslash\{\zeta^*\}\times\mathbb{C}$. Convergent series expansions for $G_\gamma(\zeta,\xi)$ can be derived in view of the relation $G_\gamma(\zeta,\xi) = \frac{\partial}{\partial\zeta}H_\gamma(\zeta,\xi)$. Namely,

$$G_\gamma(\zeta,\xi) = \sum_{k=0}^\infty\sum_{j=0}^\infty \left(\frac{j+1}{2\pi\mathrm{i}}\int_\gamma w^k z^j dz\right)\frac{1}{\zeta^{j+2}}\frac{1}{\xi^{k+1}} \tag{9}$$

for (ζ,ξ) near (∞,∞), and

$$G_\gamma(\zeta,\xi) = \sum_{k=0}^\infty\sum_{j=0}^\infty \left(\frac{-(j+1)}{2\pi\mathrm{i}}\int_\gamma w^k \frac{1}{(z-\zeta^*)^{j}} d(\frac{1}{z-\zeta^*})\right)(\zeta-\zeta^*)^j\frac{1}{\xi^{k+1}} \tag{10}$$

for (ζ,ξ) near (ζ^*,∞) for $\zeta^* \in \hat{\mathfrak{U}}^\zeta_\gamma$.

Both H_γ and G_γ are two-variable generating functions of the Wermer moments in $\mathbb{C}^2$. These generating functions are closely connected but vary slightly with respect to certain properties. For one, G_γ does not require the assumption that γ be compactly supported in $\mathbb{C}\times\mathbb{CP}^1$, unlike H_γ. Also the family of coefficients of the series expansion of $G_\gamma(\zeta,\xi)$ about (ζ^*,∞) corresponds, up to constant multiples, to the family of Wermer moments of γ with respect to the coordinates $\frac{1}{z-\zeta^*}$ and w in the affine space $(\mathbb{CP}^1\backslash\{\zeta^*\})\times\mathbb{C}$. In these ways G_γ is more tuned to $\mathbb{CP}^1\times\mathbb{CP}^1$ than H_γ is. However H_γ has the helpful feature of being a Cauchy-type integral.

Should γ be compactly supported in $(\mathbb{CP}^1\backslash\{\zeta^*\})\times\mathbb{CP}^1$, but not in $\mathbb{C}\times\mathbb{CP}^1$, then a suitable substitute for H_γ is

$$\int_{\zeta^*}^{\zeta} G_\gamma(\zeta',\xi)\, d\zeta' = \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{1}{\xi-w}\frac{(\zeta-\zeta^*)\, dz}{(z-\zeta)(z-\zeta^*)}. \tag{11}$$

This has the series expansion

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{-1}{2\pi \mathrm{i}}\int_\gamma w^k \frac{1}{(z-\zeta^*)^j} d(\frac{1}{z-\zeta^*})\right)(\zeta-\zeta^*)^{j+1}\frac{1}{\xi^{k+1}}, \tag{12}$$

for (ζ,ξ) near (ζ^*,∞) if $\operatorname{spt}\gamma \Subset \mathbb{CP}^1\backslash\{\zeta^*\}\times\mathbb{C}$.

These generating functions can be used to express characterizations for the boundaries of holomorphic 1-chains within $\mathbb{CP}^1\times\mathbb{CP}^1$ and within $\mathbb{CP}^1\times\mathbb{C}$, as is demonstrated in [9]. We reiterate these results below, incorporating some minor refinement reflecting the mentioned compatibility of G_γ with $\mathbb{CP}^1\times\mathbb{CP}^1$.

Theorem 3.1. *Let γ be a closed rectifiable 1-current in $\mathbb{CP}^1\times\mathbb{CP}^1$ with support satisfying condition A_1. Let $\zeta^*\in\mathfrak{U}^\zeta_\gamma$ and $\xi^*\in\mathfrak{U}^\xi_\gamma$. Then γ bounds a holomorphic 1-chain of finite mass within $\mathbb{CP}^1\times\mathbb{CP}^1$ if and only if there exist $R\in\mathcal{O}_{\xi^*}(\zeta)$ and $S\in\mathcal{O}_{\zeta^*}(\xi)$, neither identically zero, such that*

$$G_\gamma(\zeta,\xi) = \frac{\partial}{\partial\zeta}\left(\frac{R_\xi(\zeta,\xi)}{R(\zeta,\xi)}+\frac{S_\xi(\zeta,\xi)}{S(\zeta,\xi)}\right) \qquad \textit{near } (\zeta^*,\xi^*). \tag{13}$$

Theorem 3.2. *Let γ be a closed rectifiable 1-current compactly supported in $\mathbb{CP}^1\times\mathbb{C}$ with support satisfying condition A_1. Let $\zeta^*\in\mathfrak{U}^\zeta_\gamma$. Then γ bounds a holomorphic 1-chain of finite mass within $\mathbb{CP}^1\times\mathbb{C}$ if and only if there exists $S\in\mathcal{O}_{\zeta^*}(\xi)\cap\mathcal{O}^*_{(\zeta^*,\infty)}$ such that*

$$G_\gamma(\zeta,\xi) = \frac{\partial}{\partial\zeta}\left(\frac{S_\xi(\zeta,\xi)}{S(\zeta,\xi)}\right) \qquad \textit{near } (\zeta^*,\infty). \tag{14}$$

Remark: The existence of the decomposition (13) for a particular R and S is equivalent to the existence of a holomorphic 1-chain V bounded by γ that agrees with the divisor of R^{-1} near $w=\xi^*$ and with the divisor of S near $z=\zeta^*$, modulo components contained in $w=\xi^*$ and $z=\zeta^*$.

Due to this remark, taking Theorem 3.1 with $R=1$ would yield Theorem 3.2. Likewise, this remark with $R=1$ and $S=1$ shows that γ bounds a holomorphic 1-chain within $\mathbb{C}^2$ if and only if $G_\gamma(\zeta,\xi)=0$ near (∞,∞). This latter condition is equivalent to the vanishing moment condition, which can be seen by using the series decomposition of G_γ in (9). This is the "correspondence principle" relationship to the affine result.

So G_γ is a suitable vehicle for phrasing parallel characterizations of the boundaries of holomorphic 1-chains within $\mathbb{CP}^1\times\mathbb{CP}^1$, $\mathbb{CP}^1\times\mathbb{C}$, and $\mathbb{C}^2$. In the next two subsections we use G_γ to produce parallel characterizations within the holomorphic line bundles over $\mathbb{CP}^1$.

3.1. Negative line bundles over $\mathbb{CP}^1$.

Theorem 3.3. *Let γ be a closed rectifiable 1-current compactly supported in $U_0 \times \mathbb{C}$ of $\mathcal{O}(-n)$, $n > 0$, with support satisfying condition A_1. Then γ bounds a holomorphic 1-chain of finite mass within $\mathcal{O}(-n)$ if and only if there exists $S \in \mathcal{O}_\infty(\xi) \cap \mathcal{O}^*_{(\infty,\infty)}$ such that*

$$G_\gamma(\zeta, \xi) = \frac{\partial}{\partial \zeta}\left(\frac{S_\xi(\zeta,\xi)}{S(\zeta,\xi)}\right) \qquad \text{near } (\infty, \infty), \tag{15}$$

and one of the following equivalent supplemental conditions hold:

(1) $\frac{1}{\zeta^n}\frac{S_\xi}{S}(\zeta, \xi/\zeta^n) \in \frac{1}{\xi^2}\mathcal{O}_{(\infty,\infty)}$,
(2) $\frac{S_\xi}{S}$ *has a Taylor series at* (∞, ∞) *of the form* $\sum_{k=1}^\infty f_k(\zeta)\frac{1}{\xi^{k+1}}$ *with* $\zeta^{nk} f_k(\zeta) \in \mathcal{O}_\infty$ *for* $k \geq 1$, *or*
(3) $\frac{1}{2\pi \mathrm{i}}\int_\gamma w^k z^j \, dz = 0$ *for* $j, k \geq 0$ *such that* $j \leq nk - 2$.

Proof. By Theorem 3.2, its following remark, and Lemma 2.5, it follows that γ bounds a holomorphic 1-chain of finite mass within $\mathcal{O}(-n)$ if and only the condition of Theorem 3.2 holds along with supplemental condition 1. It only remains to show that the mentioned supplemental conditions are equivalent when the representation (15) holds.

As $\frac{S_\xi}{S} \in \frac{1}{\xi^2}\mathcal{O}_{(\infty,\infty)}$, we may express $\frac{S_\xi}{S}$ using the series expansion $\sum_{k=1}^\infty f_k(\zeta)\frac{1}{\xi^{k+1}}$ for unique $f_k \in \mathcal{O}_\infty$. Since $\frac{1}{\zeta^n}\frac{S_\xi}{S}(\zeta, \xi/\zeta^n) = \sum_{k=1}^\infty f_k(\zeta)\zeta^{nk}\frac{1}{\xi^{k+1}}$, supplemental conditions 1 and 2 are equivalent.

By considering the Taylor series expansion of G_γ given in (9), it holds that 2 and 3 are equivalent as supplemental conditions. □

We refer to the collection of integrals in condition 3 as the *Wermer moments of γ in $\mathcal{O}(-n)$*. The holomorphic 1-form $\frac{1}{2\pi \mathrm{i}} w^k z^j \, dz$ on $U_0 \times \mathbb{C}$ extends to a holomorphic 1-form on $\mathcal{O}(-n)$ when $j, k \geq 0$ and $j \leq nk - 2$. So it follows directly by Stokes theorem that condition 3 in Theorem 3.3 is necessary for γ to bound a holomorphic 1-chain within $\mathcal{O}(-n)$, $n > 0$. Later examples will show that this condition is not sufficient by itself. Besides adding the rationality condition in Theorem 3.3, one can also gain sufficiency by requiring that $\operatorname{spt}\gamma$ avoid the zero section of $\mathcal{O}(-n)$. (Note: The zero section of $\mathcal{O}(-n)$ is invariant under the biholomorphic automorphisms of $\mathcal{O}(-n)$, so avoidance of the zero section is a coordinate independent restriction.)

Theorem 3.4. *Let γ be a closed rectifiable 1-current compactly supported in $U_0 \times (\mathbb{C}\backslash\{0\})$ of $\mathcal{O}(-n)$, $n > 0$, with support satisfying condition A_1. Then γ bounds a holomorphic 1-chain of finite mass within $\mathcal{O}(-n)$ if and only if $\frac{1}{2\pi \mathrm{i}}\int_\gamma w^k z^j \, dz = 0$ for $j, k \geq 0$ such that $j \leq nk - 2$*

Proof. In light of Theorem 3.3, we see that γ bounds within $\mathcal{O}(-n)$ if and only if γ bounds within $\mathcal{O}(-1)$ and the Wermer moments $\frac{1}{2\pi \mathrm{i}}\int_\gamma w^k z^j\, dz$ vanish for $j,k \geq 0$ such that $j \leq nk-2$. Thus it suffices to establish this theorem in the case $n=1$. As the forward implication also follows from Theorem 3.3, it suffices to show that γ bounds a holomorphic 1-chain within $\mathcal{O}(-1)$ when $\frac{1}{2\pi \mathrm{i}}\int_\gamma w^k z^j\, dz = 0$ for $j \geq 0$, $k \geq j+2$ along with the given support constraint on γ.

Let ψ be the birational map from $\mathcal{O}(-1)$ to $\mathbb{C}^2$ given by $(z,w) \mapsto (w, zw)$ for $U_0 \times \mathbb{C}$ and by $(z,w) \mapsto (\frac{1}{z}w, w)$ for $U_\infty \times \mathbb{C}$. Note that $\mathcal{I}(\psi) = \emptyset$ and $\mathcal{E}(\psi)$ equals the zero section in $\mathcal{O}(-1)$. It holds by Proposition 2.1 that γ bounds within $\mathcal{O}(-1)$ if and only if $\psi_*(\gamma)$ bounds within $\mathbb{C}^2$. With the given vanishing moments, it follows using integration by parts that $\int_\gamma (wz)^{k'}(w)^{j'}\, dw = \frac{-k'}{j'+k'}\int_\gamma w^{j'+k'+1}z^{k'-1}\, dz = 0$ for $j' \geq 0$, $k' \geq 0$, which implies that $\psi_*(\gamma)$ bounds within $\mathbb{C}^2$. $\square$

The following examples show that the vanishing moment condition by itself fails to be sufficient for bounding within $\mathcal{O}(-n)$, $n > 0$.

Example 1: Let Δ be the unit disk in $\mathbb{C}$ viewed as a current of integration with multiplicity $+1$. Let $q > 1$ and define $T_{n,k,q} : \mathbb{C} \to \mathcal{O}(-n)$ to be the map given by $\lambda \mapsto (\frac{1}{k^q}\frac{1}{\lambda}, \frac{1}{k^{q(n+1)}}) \in U_\infty \times \mathbb{C}$ (or $\lambda \mapsto (\frac{1}{k^q}\frac{1}{\lambda}, \frac{1}{k^q}\lambda^n) \in U_0 \times \mathbb{C}$ for $\lambda \neq 0$). Let $\gamma_k = (T_{n,k,q})_*(b\Delta)$ and $V_k = (T_{n,k,q})_*(\Delta)$. Observe that γ_k bounds V_k within $\mathcal{O}(-n)$. Let $\gamma = \sum_{k=1}^\infty \gamma_k$, which defines a rectifiable 1-current in $\mathcal{O}(-n)$ with support satisfying condition A_1. (This fact employs a mass estimate of the form $\mathbf{M}(\gamma_k) \approx C_1\frac{1}{k^q}$.) The Wermer moments of γ in $\mathcal{O}(-n)$ vanish as the same holds for each γ_k. But $V = \sum_{k=1}^\infty V_k$ does not have finite mass in $\mathcal{O}(-n)$, plus V is not a holomorphic 1-chain as it has infinitely many sheets accumulating along the zero section. If γ bounded a holomorphic 1-chain W within $\mathcal{O}(-n)$, then $W - \sum_{k=1}^N V_k$ would have support confined to $\{|w| \leq \frac{1}{(N+1)^q}\} \cap \{|wz^n| \leq \frac{1}{(N+1)^q}\}$ for all N. This implies that W equals V plus some multiple of the zero section. Thus γ does not bound a holomorphic 1-chain within $\mathcal{O}(-n)$.

(For comparison, let ψ denote the birational map from $\mathcal{O}(-1)$ to $\mathbb{C}^2$ given by $(z,w) \mapsto (w, zw)$ on $U_0 \times \mathbb{C}$, and note that $\sum_{k=1}^\infty \psi_*(V_k)$ has finite mass in $\mathbb{C}^2$ and is a holomorphic 1-chain bounded by $\psi_*(\gamma)$ within $\mathbb{C}^2$.)

Example 2: Let $\phi_j(z) = \frac{1-4^{-j}-z}{1-(1-4^{-j})z}$. Taking the principal branch of the logarithm, $-\frac{1}{\mathrm{i}}\log\phi_j(e^{\mathrm{i}\theta})$ defines an increasing real function for θ in $(0,2\pi)$. Also $|\log\phi_j(e^{\mathrm{i}\theta})| < \frac{C}{2^{j+1}}$ for θ in $(0,2\pi)$ that satisfy $|1-e^{\mathrm{i}\theta}| > \frac{5}{2^j}$, where C is an upper bound for $\frac{\log(1+z)}{z}$ on $\{|z| < \frac{1}{2}\}$. The infinite Blaschke product $B(z) = \prod_{j=1}^\infty \frac{(1-4^{-j})-z}{1-(1-4^{-j})z}$ defines a meromorphic function on $\mathbb{CP}^1 \backslash \{1\}$. Again $-\frac{1}{\mathrm{i}}\log B(e^{\mathrm{i}\theta})$ is an increasing real function for θ in $(0,2\pi)$. Using the earlier estimate, there exists a constant C_1 such that $|\log B(e^{\mathrm{i}\theta})| \leq C_1\left(1-\log|1-e^{\mathrm{i}\theta}|\right)$ for all θ in $(0,2\pi)$. Let $f(z) = B(z)\sqrt{1-z}$, which defines a two-valued meromorphic function on $\mathbb{CP}^1 \backslash \{1\}$.

Define $f(1) = 0$, which makes both branches of f continuous over $\overline{\Delta}$. We point out that $\int_{b\Delta} |\frac{\partial}{\partial\theta} f(e^{\mathrm{i}\theta})|\, d\theta \le \int_{b\Delta} \frac{1}{2\sqrt{|1-e^{\mathrm{i}\theta}|}}\, d\theta + \int_{b\Delta} \sqrt{|1-e^{\mathrm{i}\theta}|}\frac{\partial}{\partial\theta}\left(\frac{1}{\mathrm{i}}\log B(e^{\mathrm{i}\theta})\right)\, d\theta < \infty$.

Let $T(z) = (\frac{1-z}{f(z)}, f(z))$ for z in $\overline{\Delta}\backslash\{1\}$, taking the branch of f such that $f(0)$ is positive. Define γ to be the image of $T|_{b\Delta}$, taken with appropriate orientation, in $U_0 \times \mathbb{C}$ of $\mathcal{O}(-1)$. The finiteness of the integral mentioned at the end of the previous paragraph can be used to show that γ has finite mass and so defines a rectifiable 1-current. Let $\tilde{T}(z) = \psi \circ T(z) = (f(z), 1-z)$. Observe that the image of $\tilde{T}|_\Delta$ corresponds to a holomorphic 1-chain in $\mathbb{C}^2 \backslash \operatorname{spt} \psi_*\gamma$ and is bounded by $\psi_*\gamma$ within $\mathbb{C}^2$. (By Wirtinger's Inequality and the isoperimetric inequality, one can demonstrate that it has finite mass.) So the Wermer moments of $\psi_*\gamma$ in $\mathbb{C}^2$ vanish, which implies the same for the Wermer moments of γ in $\mathcal{O}(-1)$. However γ does not bound within $\mathcal{O}(-1)$. If it did bound, it would necessarily bound the image of $T|_\Delta$, but this has infinitely many intersections with $z = \zeta$ for $|\zeta| > 1$.

Remark: As noted in Proposition 2.1 a birational map (or more generally a bimeromorphic map) $\psi : X \dashrightarrow Y$ bijectively maps the collection of boundaries of holomorphic 1-chains in X that avoid $\mathcal{I}(\psi) \cup \mathcal{E}(\psi)$ to the collection of boundaries of holomorphic 1-chains within Y that avoid $\mathcal{I}(\psi^{-1}) \cup \mathcal{E}(\psi^{-1})$. The previous examples demonstrate that this proviso regarding avoidance of the indeterminacy and exceptional sets cannot be discarded in the fundamental case of a blow-down map.

The existence of a $S \in \mathcal{O}_\infty(\xi) \cap \mathcal{O}^*_{(\zeta^*,\infty)}$ satisfying (14) serves as a type of finiteness condition on the number of sheets permitted near $z = \zeta^*$. This explains why the addition of this condition excludes the pathological behavior arising in the previous examples. Furthermore this rationality condition is readily checkable with selected bounds on degree, as shown in the following theorem and remarks.

Theorem 3.5. *Let γ be a closed rectifiable 1-current compactly supported in $U_0 \times \mathbb{C}$ of $\mathcal{O}(-n)$, $n \ge 0$, with support satisfying condition A_1. Let ζ^* be in the component of $\mathcal{U}^\zeta_\gamma$ containing ∞. Then γ bounds a holomorphic 1-chain V of finite mass within $\mathcal{O}(-n)$ such that the positive intersections of V with $z = \zeta^*$ have total degree at most M and negative intersections have total degree at most N if and only*

1. *$\frac{1}{2\pi\mathrm{i}} \int_\gamma w^k z^j\, dz = 0$ for $j, k \ge 0$ such that $j \le nk - 2$, and*
2. *there exists a rational function $C(\xi) = \frac{\sum_{k=0}^M a_k \xi^{-k}}{\sum_{k=0}^N b_k \xi^{-k}}$, with $a_k, b_k \in \mathbb{C}$ and both a_0 and b_0 non-zero such that $H := \int_{\zeta^*}^{\zeta} G_\gamma(\zeta', \xi)\, d\zeta' + \frac{C_\xi(\xi)}{C(\xi)}$ is the ξ-logarithmic derivative of a function rational in terms of $\frac{1}{\xi}$ with numerator degree at most M and denominator degree at most N (i.e. H satisfies the condition of Proposition 2.4 with M and N as given.)*

Proof. If we assume that γ bounds a holomorphic 1-chain within $\mathcal{O}(-n)$ satisfying the desired degree constraint near $z = \zeta^*$, then the Wermer moment condition 1 follows from Theorem 3.3. Furthermore we deduce that there exists a $S \in \mathcal{O}_{\zeta^*}(\xi) \cap \mathcal{O}_{(\zeta^*,\infty)}$ having numerator degree at most M and denominator degree at most N (in terms of $\frac{1}{\xi}$) such that $G_\gamma(\zeta,\xi) = \frac{S_\xi(\zeta,\xi)}{S(\zeta,\xi)}$ using Theorem 3.2 and the following remark. Setting $C(\xi) = S(\xi,\zeta^*)$, we get that $\frac{S_\xi(\zeta,\xi)}{S(\zeta,\xi)} = \int_{\zeta^*}^{\zeta} G_\gamma(\zeta',\xi)\, d\zeta' + \frac{C_\xi(\xi)}{C(\xi)}$.

For the reverse implication, note that condition 2 implies that $G_\gamma(\zeta,\xi) = \frac{\partial}{\partial\zeta} H$ with H satisfying Proposition 2.4. Thus by Theorem 3.2 and the following remark, we obtain that γ bounds a holomorphic 1-chain V within $\mathcal{O}(0)$ such that V satisfies the desired degree specifications near $z = \zeta^*$. Since γ bounds within $\mathcal{O}(0)$ and the Wermer moments for $\mathcal{O}(-n)$ vanish, we obtain via Theorem 3.3 that γ bounds some holomorphic 1-chain W within $\mathcal{O}(-n)$. It holds that $V - W$ is a true holomorphic 1-chain with compact support in $\mathcal{O}(0)$, thus $V - W$ is a linear combination of horizontal planes, i.e. varieties having the form $\mathbb{CP}^1 \times \{w_j\}$. The germs of $V - W$ near $z = \zeta^*$ (besides any in the zero section) must be contained in V. Thus we may subtract the corresponding horizontal planes from V which preserves dV and does not increase the degree of the positive and negative intersections of V with $z = \zeta^*$. Finally, by suitable addition of multiples of the zero section to W, we see that we may assume that $W = V$, which shows that γ bounds V within $\mathcal{O}(-n)$.

□

Remarks:

(1) In the case $n > 0$ and $\zeta^* = \infty$, the theorem still holds if we fix $C(\xi) = 1$ in 2. With $C(\xi)$ thus fixed, condition 2 corresponds to a differential condition (i.e. Proposition 2.4) on $\int_{\zeta^*}^{\zeta} G_\gamma(\zeta',\xi)\, d\zeta'$.

(2) In any case, fixing $C(\xi) = 1$ in condition 2 does not disrupt the reverse implication of Theorem 3.5. Now define condition 2' to be condition 2 with $C(\xi) = 1$ and with M and N each replaced with $M+N$. A slight alteration of the proof above (namely using the ξ-logarithmic derivative of $\frac{S(\zeta,\xi)}{S(\zeta^*,\xi)}$ for H) shows that the forward implication of Theorem 3.5 holds when using condition 2' in place of 2. Therefore γ bounds within $\mathcal{O}(-n)$, $n \geq 0$, if and only if the Wermer moments vanish and $H = \int_{\zeta^*}^{\zeta} G_\gamma(\zeta',\xi)\, d\zeta'$ satisfies the differential condition given by Proposition 2.4 for some M and N.

(3) If γ has no horizontal portions, then the theorem also holds for any $\zeta^* \in \mathfrak{U}^{\zeta}_{\gamma}$.

3.2. **Positive line bundles over $\mathbb{CP}^1$.**

Theorem 3.6. *Let γ be a closed rectifiable 1-current in $U_0 \times \mathbb{C}$ of $\mathcal{O}(n)$, $n > 0$, with support satisfying condition A_1. Let $\zeta^* \in \mathfrak{U}^{\zeta}_{\gamma}$. Then γ bounds a holomorphic 1-chain of finite mass within $\mathcal{O}(n)$ if and only if there exist $R \in \mathcal{O}_\infty(\zeta)$ and $S \in \mathcal{O}_{\zeta^*}(\xi)$, neither identically zero, such that*

$$G_\gamma(\zeta,\xi) = \frac{\partial}{\partial\zeta}\left(\frac{R_\xi(\zeta,\xi)}{R(\zeta,\xi)} + \frac{S_\xi(\zeta,\xi)}{S(\zeta,\xi)}\right) \qquad \text{near } (\zeta^*,\infty), \tag{16}$$

and one of the following equivalent supplemental conditions hold:

(1) $\zeta^n \frac{R_\xi}{R}(\zeta, \xi\zeta^n) \in \frac{1}{\xi^2}\mathcal{O}_{(\infty,\infty)}$ *and* $\frac{R_\xi}{R}(\zeta,\xi) \in \frac{1}{\xi^2}\mathcal{O}_{(\zeta_0,\infty)}$ *for all* $\zeta_0 \neq \infty$*, or*

(2) $\frac{R_\xi}{R}$ *has a series development at* $\xi = \infty$ *of the form* $\sum_{k=1}^{\infty} f_k(\zeta)\frac{1}{\xi^{k+1}}$ *with each* $f_k(\zeta)$ *being a polynomial with degree at most* nk*.*

(Note: Since $R \in \mathcal{O}_\infty(\zeta)$, it defines germs in $\mathcal{M}_{(\zeta_0,\infty)}$ for any ζ_0.)

Proof. As may be seen from the construction in the proof of Theorem 3.1 one can arbitrarily select $\zeta^* \in \mathcal{U}_\gamma^\zeta$ without modifying R. Since the supplemental conditions depend only on R, without loss of generality we may assume that $\zeta^* = \infty$.

By Theorem 3.1, the remark following it, Lemma 2.5, and Proposition 2.2, it holds that γ bounds a holomorphic 1-chain of finite mass within $\mathcal{O}(n)$ if and only if the condition of Theorem 3.1 holds along with the following, which we term supplemental condition 0.

(0) $\zeta^n \frac{S_\xi}{S}(\zeta, \xi\zeta^n) \in \frac{1}{\xi^2}\mathcal{O}_{(\infty,\infty)}$ and $\frac{R_\xi}{R}(\zeta,\xi) \in \frac{1}{\xi^2}\mathcal{O}_{(\zeta_0,\infty)}$ for all $\zeta_0 \neq \infty$

When the condition of Theorem 3.1 holds we may integrate (13) to obtain that there exists a $m \in \mathbb{Z}$ and a $f(\xi) \in \frac{1}{\xi^2}\mathcal{O}_\infty$ such that

$$H_\gamma(\zeta,\xi) = \frac{R_\xi}{R}(\zeta,\xi) + \frac{S_\xi}{S}(\zeta,\xi) + \frac{m}{\xi} + f(\xi) \text{ near } (\infty,\infty). \tag{17}$$

(To see why the later two terms have their form, it is useful to note that $H_\gamma(\infty,\xi) = 0$ and that $\frac{h_\xi(\xi)}{h(\xi)}$ resides in $\frac{1}{\xi}\mathbb{Z} + \frac{1}{\xi^2}\mathcal{O}_\infty$ for any h in $\mathcal{M}_\infty$.) Replace $R(\zeta,\xi)$ with $R(\zeta,\xi)\exp\left[\int_\infty^\xi f(\xi')\, d\xi'\right]$, which does not change the assumptions on R but removes the term $f(\xi)$ from above. Furthermore this "folding in of f" does not affect whether each individual supplemental condition holds or not. Thus

$$\zeta^n H_\gamma(\zeta,\xi\zeta^n) = \zeta^n \frac{R_\xi}{R}(\zeta,\xi\zeta^n) + \zeta^n \frac{S_\xi}{S}(\zeta,\xi\zeta^n) + \frac{m}{\xi} \text{ near } (\infty,\infty). \tag{18}$$

If we assume supplemental condition 0, then each term, save $\frac{m}{\xi}$, is in $\frac{1}{\xi^2}\mathcal{O}_{(\zeta_0,\infty)}$ for some finite ζ_0 close to ∞, thus $m = 0$. Therefore $\zeta^n \frac{R_\xi}{R}(\zeta,\xi\zeta^n) \in \frac{1}{\xi^2}\mathcal{O}_{(\infty,\infty)}$, which implies supplemental condition 1.

Conversely suppose that supplemental condition 1 holds. Replacing $S(\zeta,\xi)$ with $\xi^m S(\zeta,\xi)$ allows us to assume that $m = 0$. Then $\zeta^n \frac{S_\xi}{S}(\zeta,\xi\zeta^n) \in \frac{1}{\xi^2}\mathcal{O}_{(\infty,\infty)}$ follows from (18) and so supplemental condition 0 holds.

For $R \in \mathcal{O}_\infty(\zeta)$, $\frac{R_\xi}{R}$ has the series development $\sum_{k=0}^{\infty} f_k(\zeta)\frac{1}{\xi^{k+1}}$ for uniquely defined $f_k(\zeta) \in \mathbb{C}(\zeta)$ with convergence occurring for ξ near ∞ and those ζ such that $R(\zeta,\infty)$ is finite and nonzero. Note that $\zeta^n \frac{R_\xi}{R}(\zeta,\xi\zeta^n)$ has the series expansion $\sum_{k=0}^{\infty} \frac{f_k(\zeta)}{\zeta^{nk}}\frac{1}{\xi^{k+1}}$. Thus supplemental condition 1 is equivalent to saying that $f_0 = 0$ and that for $k \geq 1$, $f_k \in \mathbb{C}[\zeta]$ and $f_k \in \zeta^{nk}\mathbb{C}[\frac{1}{\zeta}]$. Thus supplemental conditions 1 and 2 are equivalent.

□

As the following reveals, one may use supplemental condition 2 in Theorem 3.6 to describe $\frac{R_\xi}{R}$ without any need to establish rationality for R.

Theorem 3.7. *Let γ be a closed rectifiable 1-current in $U_0 \times \mathbb{C}$ of $\mathcal{O}(n)$, $n > 0$, with support satisfying condition A_1. Let $\zeta^* \in \mathfrak{U}^\zeta_\gamma$. Then γ bounds a holomorphic 1-chain of finite mass within $\mathcal{O}(n)$ if and only if there exist $S \in \mathcal{O}_{\zeta^*}(\xi)$ not identically zero, polynomials $f_k(\zeta) \in \mathbb{C}[\zeta]$ of degree at most nk for $k \geq 1$, and an integer m such that*

$$\int_{\zeta^*}^{\zeta} G_\gamma(\zeta', \xi)\, d\zeta' = \frac{m}{\xi} + \sum_{k=1}^{\infty} f_k(\zeta) \frac{1}{\xi^{k+1}} + \frac{S_\xi(\zeta, \xi)}{S(\zeta, \xi)} \qquad \text{near } (\zeta^*, \infty). \tag{19}$$

Proof. The forward implication follows by Theorem 3.6 and some elements of its proof. So it suffices to prove the reverse implication. Without loss of generality, we may assume that S is in (N2) form, and thus we assume that $m = 0$.

Consider the case $\zeta^* \neq \infty$. Take the divisor of S in a neighborhood of $z = \zeta^*$ to define a holomorphic 1-chain V_S bounded by a finite, smooth, real 1-chain Γ_S within $U_0 \times \mathbb{C}$. Let $\Gamma = \gamma - \Gamma_S$. Near (ζ^*, ∞), $G_{\Gamma_S}(\zeta, \xi) = \frac{\partial}{\partial \zeta} \frac{S_\xi}{S}(\zeta, \xi)$ and so $G_\Gamma(\zeta, \xi) = \sum_{k=1}^{\infty} f_k'(\zeta) \frac{1}{\xi^{k+1}}$. By considering the series decomposition of G_Γ at (ζ^*, ∞), as in (10), it holds that $\frac{1}{2\pi \mathrm{i}} \int_\Gamma \left(\frac{w}{(z-\zeta^*)^n}\right)^k \left(\frac{1}{z-\zeta^*}\right)^{j-nk} d\left(\frac{1}{z-\zeta^*}\right) = 0$ for $k \geq 0$ and $j \geq nk$. Since $\frac{1}{z-\zeta^*}$ and $\frac{w}{(z-\zeta^*)^n}$ provide affine coordinates for $\mathcal{O}(n) \backslash \{z = \zeta^*\}$, it follows that Γ bounds a holomorphic 1-chain V_S' within $\mathcal{O}(n) \backslash \{z = \zeta^*\}$. In conclusion, γ bounds the holomorphic 1-chain $V := V_S + V_S'$ within $\mathcal{O}(n)$.

For the case $\zeta^* = \infty$, simply perturb ζ^* to be a finite value near ∞ and use the preceding argument.

□

Note that $\mathcal{O}(1)$ is biholomorphic to $\mathbb{CP}^2 \backslash (0:0:1)$ via the map $(z, w) \mapsto (1 : z : w)$ on $U_0 \times \mathbb{C}$ and $(z, w) \mapsto (\frac{1}{z} : 1 : w)$ on $U_\infty \times \mathbb{C}$. So these results have connections to $\mathbb{CP}^2 \backslash (0:0:1)$ which has also been studied in [8] as a special, though foundational, case in connection with work pertaining to $\mathbb{CP}^m \backslash \mathbb{CP}^{m-2}$. To discuss this further, we present the following corollary which suitably mirrors Theorem 8.3 of [8].

Corollary 3.8. *With the initial assumptions of Theorem 3.7, γ bounds a holomorphic 1-chain V of finite mass within $\mathcal{O}(n)$, $n \geq 0$, with the positive intersections of V with $z = \zeta^*$ having degree at most M and the negative intersections having degree at most N if and only there exist polynomials $f_k(\zeta)$, such that each f_k has degree at most nk and $H := \int_{\zeta^*}^{\zeta} G_\gamma(\zeta', \xi)\, d\zeta' - \sum_{k=1}^{\infty} f_k(\zeta) \frac{1}{\xi^{k+1}}$ satisfies the condition of Proposition 2.4 with M and N as given.*

In [8], Theorem 8.3, along with Note 8.5, largely corresponds to Corollary 3.8 here in the special case that $n = 1$, $N = 0$, and γ is a C^1 chain. (The case

$N = 0$ corresponds to taking S to be a polynomial in ξ, in which case $\frac{S_\xi}{S}$ is a generating function of the Newton hierarchy used in [8].) So the case $n = 1$ of Corollary 3.8 here provides an extension of Theorem 8.3 that handles both negative intersections and positive intersections between the bounded holomorphic 1-chain and the line $z = \zeta^*$. The discussion in Section 3 of [8], considered more broadly, offers commentary relevant to Corollary 3.8 in the case $N = 0$. For $N > 0$, that commentary can be augmented by the results and observations given presently in Subsection 2.2.

References

[1] Dinh T. C., Enveloppe polynomiale d'un compact de longueur finie et chaînes holomorphes à bord rectifiable, Acta Math. 180 (1998) 31-67

[2] Dolbeault P., Henkin G., Surfaces de Riemann de bord donne dans $\mathbb{CP}^n$, Contributions to complex analysis and analytic geometry, Aspects of Math., Vieweg, 26 (1994) 163-187

[3] Federer H., Geometric Measure Theory, Die Grundlehren der math. Wissenschaften, Band 153, Springer-Verlag, New York, (1969)

[4] Fischer G., Complex Analytic Geometry, Lecture Notes in Mathematics, Vol 538, Springer-Verlag, New York, (1976)

[5] Fritzsche K., Grauert H., From Holomorphic Functions to Complex Manifolds, Graduate Texts in Mathematics, Vol 213, Springer-Verlag, New York, (2002)

[6] Harvey, R., Holomorphic chains and their boundaries, Proc. Symp. Pure Math., 30 (1977), 309-382

[7] Harvey R., Lawson B., On boundaries of complex analytic varieties, I, Ann. of Math. 102 (1975) 223-290

[8] Harvey R., Lawson B., Boundaries of varieties in projective manifolds, J. Geom. Anal., 14 (2004), 673-695

[9] Walker R. A., Characterization of boundaries of holomorphic 1-chains within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ and $\mathbb{C} \times \hat{\mathbb{C}}$, J. Geom. Anal., 18 (2008), 1159-1170

[10] Walker R. A., Extended shockwave decomposability related to boundaries of holomorphic 1-chains within $\mathbb{CP}^2$, Indiana Univ. Math. J., 57(2008) 1133-1172

[11] Wermer J., The hull of a curve in $\mathbb{C}^n$, Ann. of Math. 68 (1958) 550-561

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